Calculus

Real Numbers, limit, continuity and differentiability of single variable functions

Real numbers, functions of a real variable, limits, continuity, differentiability, mean value theorem, Taylor's theorem with remainders, indeterminate forms, maxima and minima, asymptotes; curve tracing; functions of two or three variables: limits, continuity, partial derivatives, maxima and minima, Lagrange's method of multipliers, Jacobian.

Riemann's definition of definite integrals; indefinite integrals; infinite and improper integrals; double and triple integrals (evaluation techniques only); areas, surface and volumes.

Natural no.

Whole No.

Integers

Rational no.

Irrational no.

Properties of real numbers

If a, b, and c are any given real numbers, then order axiom says-

1. Law of trichotomy- Either $a > b$, $a = b$ or $a < b$
2. Law of transitivity- If $a > b$ and $b > c$, then $a > c$
3. If $a > b$, then $a + c > b + c$
4. If $a > b$ and $c > 0$, then $ac > bc$
5. If $a > b$ and $c < 0$, then $ac < bc$

Set, Interval, boundedness

1.5.3. Theorem.

If S is a non-empty set of reals which is bounded above, then a real number u is supremum (i.e., least upper bound) of S iff

(i) $x \leq u$ *for all* $x \in S$

(ii) *for each* $\varepsilon > 0$*, there exists atleast one* $x \in S$ *such that* $x > u - \varepsilon$

Uniqueness of Supremum

Theorem. *Least upper bound of a set, if it exists, is unique.*

.2. Theorem.

The greatest member of a set, if it exists, must be the supremum of the set.

Properties of real numbers

Completeness Axiom- Every non-empty subset of reals which is bounded above has a real number as its supremum.

Archimedean property of Reals- If x and y are two positive real numbers, then there exists a natural number n such that nx>y.

Infimum of S<= Supremum of S

Absolute Value of Real Numbers

The absolute value of a real number a, denoted by $|a|$, is defined as a if $a > 0$, $-a$ if $a < 0$, and 0 if $a = 0$.

Properties of Absolute Value

1. $|ab| = |a|\ |b|$ or $|abc \ldots m| = |a|\ |b|\ |c| \ldots |m|$

2. $|a + b| < |a| + |b|$ or $|a + b + c + \ldots + m| < |a| + |b| + |c| + \ldots |m|$

3. $|a - b| > |a| - |b|$

1.11. NEIGHBOURHOOD OF A POINT

A set $S \subseteq \mathbf{R}$ is said to be a neighbourhood of a point $p \in \mathbf{R}$, if there exists an open interval $(p - \varepsilon, p + \varepsilon)$ for some $\varepsilon > 0$ such that $p \in (p - \varepsilon, p + \varepsilon) \subseteq S$.

This neighbourhood of point p is called a symmetric neighbourhood of p with radius ε.

We can also define neighbourhood of a point as follows :

A subset S of $\mathbf{R}$ is a neighbourhood of a point $p \in \mathbf{R}$, if there exists an open interval (a, b) such that $p \in (a, b) \subseteq S$.

Any interval
Finite set
N, I, R, Q, phi

Deleted nbd

superset of a neighbourhood of a point p is also a nbd of p.

If A and B are two nbds Of a point p, then their intersection is again a nbd of p.

1.15. INTERIOR POINT OF A SET

If A is neighbourhood of a point x, then point x is called an interior point of A.

We may also define interior point of a set as follows :

Let A be a subset of $\mathbf{R}$. *The point* $x \in A$ *is said to be an* **interior point** *of A if there exists an open interval* (a, b) *such that* $x \in (a, b) \subseteq A$

Open set

1. Union of an arbitrary family of open Sets is an open set.
2. Intersection of a finite number of open sets is an open set.
3. Intersection of arbitrary family of open sets may or may not be an open set.

Interior of a set is an open set.

1.18. CLOSED SETS

Let A be any subset of $\mathbf{R}$. Then set A is said to be a **closed set** iff complement of A is an open set *i.e.*, A is closed iff $A^c = \mathbf{R} - A$ is open.

Illustrations.

1. The set $\mathbf{R}$ of real numbers is a closed set as its complement $\mathbf{R}^c = \mathbf{R} - \mathbf{R} = \phi$ is an open set.
2. Set ϕ is a closed set as its complement $\phi^c = \mathbf{R} - \phi = \mathbf{R}$ is an open set.
3. Every closed interval $[a, b]$ is a closed set. The complement of $[a, b]$ is $(-\infty, a) \cup (b, \infty)$ which is an open set as it is a union of two open sets.
4. The closed ray $(-\infty, a]$ is a closed set as its complement (a, ∞) is an open set.

5. [a,b]

6. A finite set

7. N

8. Z

9. Q

10. R-N

11. R-Q

12. A = (2,3) U {4,5}

1. Intersection of an arbitrary family of closed sets is closed

The union of finite number of closed sets is a closed set.

The Arbitrary union of closed Sets may not be a closed set.

1.20. LIMIT POINT OF A SET

$p \in S$

A real number p is said to be a limit point or accumulation point of a set $S \subseteq \mathbf{R}$ if every neighbourhood of p contains at least one point of S different from p.

Another definition of limit point.

$$((p-\epsilon, p+\epsilon) - \{p\}) \cap S \neq \phi$$

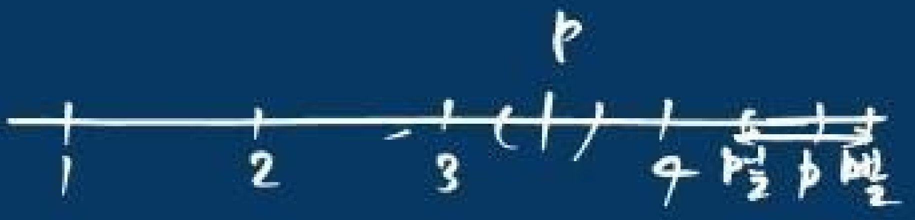
A real number p is said to be a limit point of a set $S \subseteq \mathbf{R}$ if every neighbourhood of p contains infinitely many points of S.

nbd $\equiv$ interval on Real line

Write the set of limit point of following sets =

1. $(Q)' = R$
2. $(R-Q)' = R$
3. $(N)' = \phi$
4. $(Z)' = \phi$
5. $(R)' = R$
6. $S = \{1/n, n \in N$

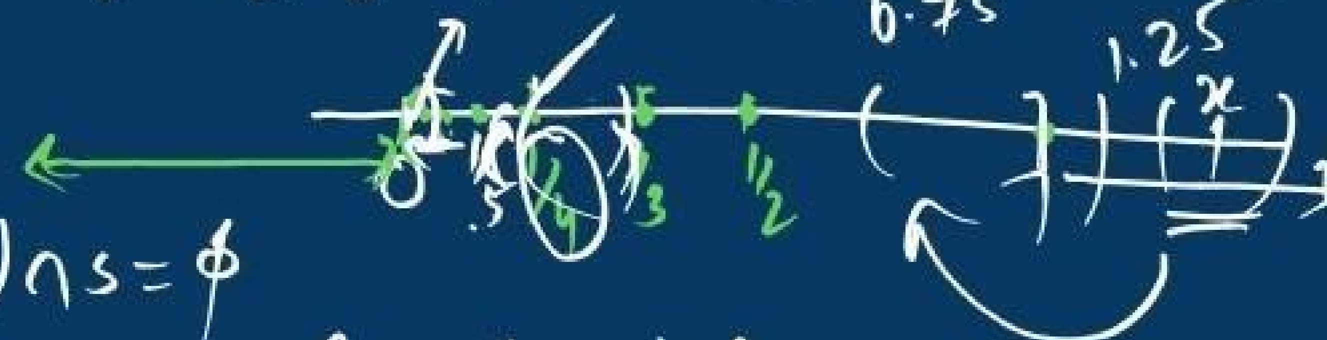
(3)



(i) If $p \in N$ let $\epsilon = \frac{1}{2}$
then $((p-\frac{1}{2}, p+\frac{1}{2}) - \{p\}) \cap N = \phi$
$\Rightarrow p \in N$ is not a limit point

(ii) If $p \in R-N$
then $((p-\epsilon, p+\epsilon) - \{p\}) \cap N = \phi$
$(N)' = \phi$

(6) $S = \{\frac{1}{n} \in N\} = \{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4} \ldots\ldots\}$



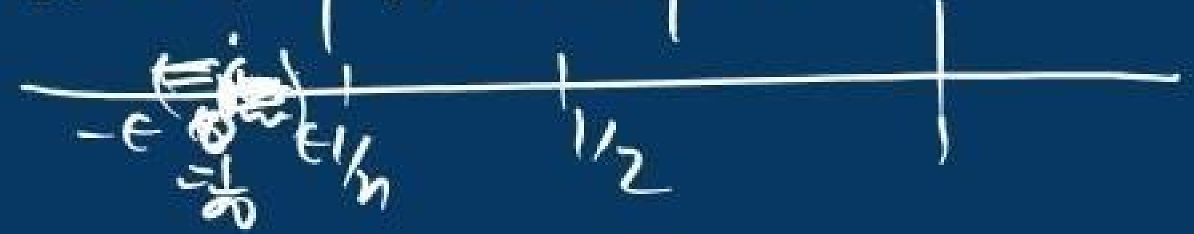
(i) Let $x \in (-\infty, 0)$
then $((x-\epsilon, x+\epsilon) - \{0\}) \cap S = \phi$
$\Rightarrow x \in (-\infty, 0)$ is not a limit point.

(ii) Let $x \in (1, \infty)$
$\Rightarrow (x-\epsilon, x+\epsilon) \cap S = \phi \Rightarrow x \in (1, \infty)$ is not a limit point

(iii) Let $x = 1$, $1 \in ((0.75, 1.25) - \{1\}) \cap S = \phi$
$\Rightarrow 1$ is not a L.P.

(iv) $x = \frac{1}{n}$ then $\left(\left(\frac{1}{n+1}, \frac{1}{n-1}\right) - \left\{\frac{1}{n}\right\}\right) \cap S = \phi$

(v) $x = 0$ then $((0-\epsilon, 0+\epsilon) - \{0\}) \cap S \neq \phi$
$\Rightarrow$ 0 is limit point of S.



1.20.1. Derived Set

The set of all limit points of a set $A \subseteq \mathbf{R}$ *is called a derived set of A and is denoted by* A' *or* $d(A)$.

Thus, $$A' = \{x \in \mathbf{R} \mid x \text{ is a limit point of } A\}$$

For example, $(2, 3)' = [2, 3]$; $[5, 6]' = [5, 6]$

$$\mathbf{N}' = \phi, \quad \mathbf{Z}' = \phi, \quad \mathbf{R}' = \mathbf{R} \text{ and } \phi' = \phi.$$

1.20.2. Isolated Point

Let A be a subset of **R**. Then $x \in \mathbf{R}$ is called an isolated point of set A, if $x \in A$ but x is not a limit point of A

1.20.3. Adherent Point

Let A be a subset of **R**. Then $x \in \mathbf{R}$ is called an adherent point of A, if every neighbourhood of x contains atleast one point of A.

i.e., $$x \in (x - \varepsilon, x + \varepsilon) \cap A \neq \phi, \text{ for all } \varepsilon > 0$$

$2 \in ((1.5, 2.5) - \{2\}) \cap (2, 3) \neq \phi$ $\rightarrow$ 2, 3 limit

Remarks :

1. In order to show that a point p is not an adherent point of a set A, it is sufficient to find a neighbourhood B of p such that $B \cap A = \phi$.

2. Students should compare the definition of limit point of a set with that of adherent point of a set. A point p is a limit point of a set A if every neighbourhood of p must contain a point of $\boxed{\text{A other than } p}$, whereas p is an adherent point of set A if every neighbourhood of p contains a point of A which can be p itself. Thus it can be easily concluded from the definitions of limit point and adherent point of a set that

   (a) **Every limit point of a set is also the adherent point of that set.**

   (b) **An adherent point may or may not be a limit point of a set**

   For example, set of natural numbers N has all its points as adherent points but this set has no limit point.

1.21. CLOSURE OF A SET

Let A be subset of **R**. Then closure of A is the set containing all points of A and all limit points of A. It is denoted by $\overline{A}$ $= A \cup A' = A \cup d(A)$

In other words, closure of A, denoted by $\overline{A}$, is the set containing all adherent points of A

Thus $\boxed{\overline{A} = A \cup d(A) \text{ or } \overline{A} = A \cup A'}$

Illustrative Example. *Find the derived sets and closure sets of the following sets :*

(i) *Open interval* (a, b)

(ii) *Closed interval* $[a, b]$

(iii) *Half open interval* $[a, b)$

(iv) *Half open interval* $(a, b]$

(i) $(a,b)' = [a,b]$

(ii) $[a,b]' = [a,b]$

(iii) $[a,b)' = [a,b]$

(iv) $(a,b] = [a,b]$

Closure $A \cup d(A)$ $(a,b) \cup [a,b] = [a,b]$

(v) $A = \left\{\frac{1}{n} : n \in \mathbf{N}\right\}$ $A' = \{0\}$

$\bar{A} = A \cup A' = \{0, \frac{1}{n}\}, n \in N$

(vi) $\mathbf{N}$, $N' = \phi$, $\bar{N} = N$

(vii) $\mathbf{Z}$, $Z' = \phi$ $\bar{Z} = Z$

(viii) $\mathbf{Q}$, $Q' = R$, $\bar{Q} = R$

(ix) $\mathbf{R}$, $R' = R$ $\bar{R} = R$

(x) $\mathbf{R} - \mathbf{N}$, $(R-N)' = R$ $\overline{R-N} = R$

(xi) $\mathbf{R} - \mathbf{Z} = R$

(xii) $\mathbf{R} - \mathbf{Q} = (R-Q)' = R$

(xiii) $\mathbf{R} - \mathbf{I} = $ (viii)

(xiv) ϕ

$\{0, \underbrace{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots}\}$

$x \in (0, \infty)$
But $x \notin N$

$\underline{R - N}$

$((x-\epsilon, x+\epsilon) - \{x\}) \cap (R-N) \neq \phi$

$x \notin N$

$\to n-1 \leq x < n$

$\epsilon < \min\{|x-(n-1)|, |n-x|\}$

$((x-\epsilon, x+\epsilon) - \{x\}) \cap R-N \neq \phi$

$x \in N$, $((x-\frac{1}{2}, x+\frac{1}{2}) - \{x\}) \cap R-N \neq \phi$

A point p is a limit point of a set A if and only if every neighbourhood of p contains infinitely many points of A. [M.D.U. 2013]

Let $A \subseteq R$

Let p is limit point of A.

N.T.P. :- Every nbd of p contains infinite points of A.

Let if possible $\exists$ a nbd $(p-\epsilon, p+\epsilon)$ of p such that it contain of finit points of A which are $p_1, p_2, \ldots p_n$

$\delta < \min\{|p-p_1|, |p-p_2|, |p-p_3|, \ldots |p-p_n|\}$

Now $(p-\delta, p+\delta)$ is also a nbd of p.

$((p-\delta, p+\delta) - \{p\}) \cap A = \phi \Rightarrow p$ is not a limit point #

Conversely $\rightarrow$ Every nbd of p contains infinite points of A.

T.S. $\rightarrow$ p is a limit

Let $(p-\epsilon, p+\epsilon)$ is one of nbds of $p \Rightarrow$ then $((p-\epsilon, p+\epsilon) - \{p\}) \cap A \neq \phi \Rightarrow p$ is a l.p.

4. BOLZANO-WEIERSTRASS THEOREM

Statement. *Every infinite bounded subset of real numbers has a limit point.*

[K.U. 2013, 11; M.D.U. 2013, 12]

Imp

Let A is an infinite bounded subset of $\mathbb{R}$.

Let $\inf A = m$ & $\sup A = M$

$T = \{x : x$ exceeds on finite members of $A\}$

$x \in T \Rightarrow \begin{cases} x \geq \text{only finite no. of elements of } A \\ \text{or} \\ x \leq \text{infinite no. of elements of } A \end{cases}$

Now $m \in T \Rightarrow T \neq \phi$

Let $y \in T$ is arbitrary

$\Rightarrow y \leq$ infinite no. of elements of A.

$\Rightarrow y \leq M \Rightarrow$ all the elements of T are $\leq M$

T is upper bounded set.

By completeness axiom of Real

m A $u-\epsilon$ M $u+\epsilon$

$m+\epsilon$

$m < x_1 < m+\epsilon$, $x_1 \in A$

$M-\epsilon \leq x_2 \leq M$, $x_2 \in A$

T has a supremum say u

then $u-\epsilon$ is not an upper bound

$\Rightarrow u-\epsilon < x < u$, $x \in T$ —— (1)

and $u+\epsilon \notin T$

$\Rightarrow u+\epsilon \geq$ infinite no. of elements of A

then $u-\epsilon < x < u < u+\epsilon$

then $(u-\epsilon, u+\epsilon)$ contains infinite point of A

$\Rightarrow u$ is a limit point of A.

$A = \{ x :$ height of x should be less then 5' $\}$

$\underbrace{y > 5'}$

1.22.7. Theorem.

Prove that the supremum of a non-empty bounded set is either the greatest member of the set or is a limit point of the set. [K.U. 2012]

Let A is a non-empty bdd set.

By completeness property of real,

A would have a supremum say u.

<u>Case I.</u> <u>If $u \in A$</u>

Now u is sup. $\Rightarrow$ $u \geq x$; $x \in A$

$\Rightarrow$ u is greatest member of A

<u>Case II</u>:- If $u \notin A$ then $\epsilon > 0$;

$\underbrace{u - \epsilon < x < u}$; $\underline{x \in A}$

$\underline{u - \epsilon < x < u < u + \epsilon}$

Now $(u-\epsilon, u+\epsilon)$ is nbd of u.

$\{ \underbrace{(u-\epsilon, u+\epsilon)}_{x} - \{u\}] \cap \underline{A} \neq \phi$

$\Downarrow$

$\Rightarrow$ u is a limit point

Compact set = A non-empty subset $A \subseteq \mathbb{R}$ is stb compact.

if it is closed & bounded

(1) $[a,b]$

(2) $[3,4] \cup [5,6]$

(3). $[3,4] \cap [4,5] = \{4\}$

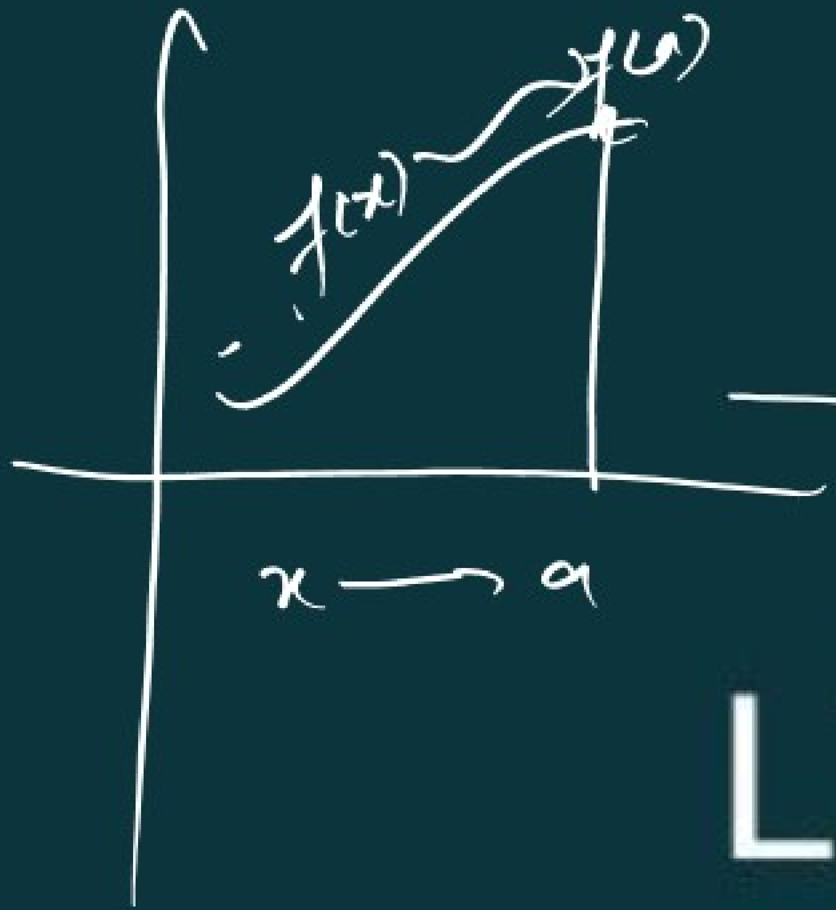
(4). Any finite set.

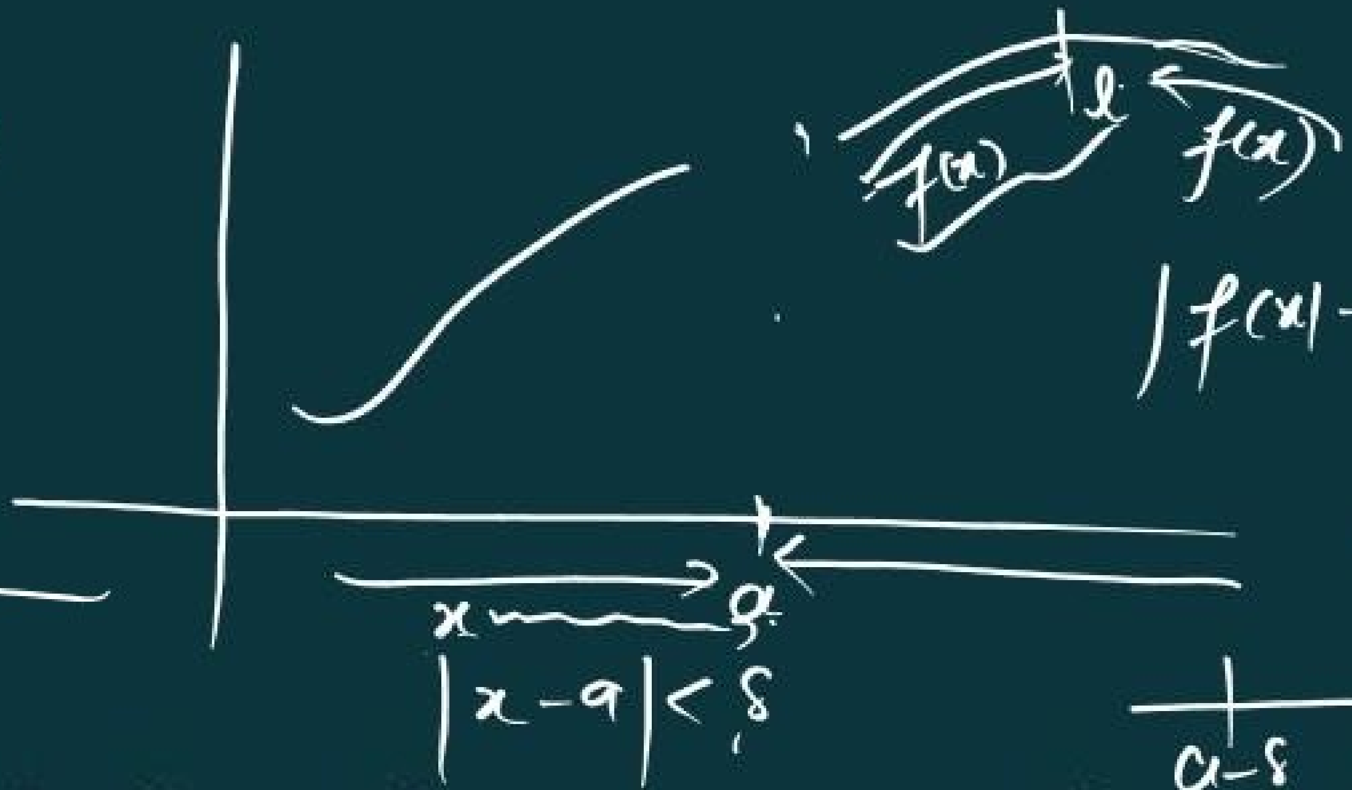
5. (a,b)

6. $\mathbb{R}$

7. $\mathbb{Q}$

8. $\mathbb{Z}$





$|f(x)-l| < \epsilon$

for $\epsilon > 0 \ \exists\ \delta > 0$ s.t. $0 < |f(x)-l| < \epsilon$ where $0 < |x-a| < \delta$

Limit, continuity and differentiability

$a < x < a+\delta$

$|x-a| < \delta$

$-\delta < x-a < \delta$

$a-\delta < x < a+\delta$

$a-\delta < x < a$

Definition 1 : Limit of a function : A function f is said to tend to limit 'l' as x tends to 'a' if for a given $\varepsilon > 0$, however small, there exists a positive real number δ (depending upon ε) such that

$$|f(x)-l| < \varepsilon \quad \text{whenever} \quad 0 < |x-a| < \delta$$

$a-h \quad a \quad a+h$

In symbols we write $\boxed{\lim_{x \to a} f(x) = l}$

Definition 2 : Left Hand limit : A function 'f' is said to tend to left limit 'l_1', as x tends to 'a' from left (i.e., through values less than a) if for given $\varepsilon > 0$, there exists $\delta_1 > 0$ such that

$$|f(x)-l_1| < \varepsilon \quad \text{whenever} \quad a-\delta_1 < x < a$$

In symbols, we write $\lim_{x \to a^-} f(x) = l_1$

L.H.L. $\lim_{x \to a^-} f(x) = \lim_{h \to 0} f(a-h)$

R.H.S $\lim_{x \to a^+} f(x) = \lim_{h \to 0} f(a+h)$

Definition 3 : Right Hand limit: A function 'f' is said to tend to right limit 'l_2' as x tends to 'a' from right (i.e., through values greater than a) if for given $\varepsilon > 0$, there exists $\delta_2 > 0$ such that

$$|f(x)-l_2| < \varepsilon \quad \text{whenever} \quad a < x < a+\delta_2$$

$\lim_{x \to a^+} f(x) = l_2$

Symbolically, we write $\lim_{x \to a^+} f(x) = l_2$

Limit of a function if exists, then it is unique.

By using $\varepsilon - \delta$ definition of limit show that $\lim_{x \to 1} \frac{x^2 - 1}{x - 1} = 2, \quad x \neq 1.$

$l = 2$
$a = 1.$

$f(x) = \frac{x^2-1}{x-1}$

$$\boxed{|x-1| < \delta}$$

Let's consider $|f(x) - l| < \epsilon$ for some $\boxed{\epsilon > 0}$

$$\Rightarrow \left| \frac{x^2-1}{x-1} - 2 \right| < \epsilon$$

$$\Rightarrow \left| \frac{x^2-1-2x+2}{x-1} \right| < \epsilon$$

$$\Rightarrow \left| \frac{(x-1)^2}{x-1} \right| < \epsilon$$

$\epsilon > 0$

$$\Rightarrow \underline{|x-1| < \epsilon = \delta}.$$

By using definition of limit, show that $\lim\limits_{x \to 0} \left(x \cos \frac{1}{x} \right) = 0$.

ℓ

$f(x) = x \cos \frac{1}{x}$.

$\ell = 0$

$a = 0$

$|x - 0| < \epsilon = \delta$.

Let's consider for some $\epsilon > 0$ $|f(x) - \ell| < \epsilon$

$\Rightarrow \left| x \cos \frac{1}{x} - 0 \right| < \epsilon$

$\Rightarrow \left| x \cos \frac{1}{x} \right| < \epsilon$

$\Rightarrow |x| \left| \cos \frac{1}{x} \right| < \epsilon$ $\left(\because \left| \cos \frac{1}{x} \right| = 0 \text{ to } 1 \right)$

$\Rightarrow$

Prove that $\lim_{x \to 0} \sin\left(\frac{1}{x}\right)$ **does not exist.**

(If two sequences are approaching to same limit then their function should also approach to same limit)

if limit exist

$$\boxed{\text{Let } x_1 \to \left\langle \frac{1}{2n\pi} \right\rangle \ \& \ x_2 \to \left\langle \frac{1}{2n\pi + \frac{\pi}{2}} \right\rangle}$$

$\lim x_1 = 0$ & $\lim x_2 = 0$

Now

$$\lim_{x_1 \to 0} f(x_1) = \lim_{x_1 \to 0} \sin\left(\frac{1}{x_1}\right)$$
$$= \lim_{x_1 \to 0} \sin(2n\pi) = 0$$

$$\lim_{x_2 \to 0} f(x_2) = \lim_{x_2 \to 0} \sin\left(\frac{1}{x_2}\right) = \lim_{x_2 \to 0} \sin\left(2n\pi + \frac{\pi}{2}\right)$$
$$= \lim_{x_2 \to 0} \sin\frac{\pi}{2} = 1$$

$(x_1)_n \to a$

$(x_2)_n \to a$

If $f(x_{1n}) \to l$

then $f(x_{2n}) \to l$

$\Rightarrow$ By sequencial approach limit of given function does not exist.

Example 7 : If $f(x) = \begin{cases} 0 & \text{if } x \text{ is irrational} \\ 1 & \text{if } x \text{ is rational} \end{cases}$ then prove that $\lim_{x \to a} f(x)$, $a \in R$ does not exist.

Let $\langle x_1 \rangle$ is a seq^n of irrational no s.t.
$\lim \langle x_1 \rangle = a$
& $\langle x_2 \rangle$ is a seq^n of rational no. s.t.
$\lim \langle x_2 \rangle = a$

Let if possible limit exists & say l
Let's consider $|f(x_1) - l| = |0 - l| = |l|$
Again consider $|f(x_2) - l| = |1 - l| = |1 - l|$
Hence limit doesn't exist.

$$\boxed{\begin{aligned} |a-b| &\geq |a| - |b| \\ |a-b| &= |a + (-b)| \leq |a| + |b| \end{aligned}}$$

Let x_1 is rational no
$f(x_1) = 1$ —— (1)
Let x_2 is irrational then, $f(x_2) = 0$ —— (2)

If possible. Let limit exists. "say l" then for given $\epsilon > 0$; $\exists\ \delta > 0$
s.t. $|f(x) - l| < \epsilon$ whenever $0 < |x-a| < \delta$ (1)
Let's consider $\epsilon = 1/2$
Now for x_1; $|f(x_1) - l| < 1/2$ —(3) & $|f(x_2) - l| < 1/2$ —(4)

1st $\boxed{|f(x_1) - f(x_2)| = |1 - 0| = 1}$ (by using (1) & (2)) (5)

2nd $|f(x_1) - f(x_2)| = |(f(x_1) - l) + (f(x_2) - l)| \leq |f(x_1) - l| + |f(x_2) - l| < 1/2 + 1/2 = 1$
$\Rightarrow |f(x_1) - f(x_2)| < 1$
$\Rightarrow 1 < 1$ #

Algebra of limits

Theorem 2.1 : Let f and g be two functions such that $\lim\limits_{x \to a} f(x) = l$ and $\lim\limits_{x \to a} g(x) = m$ then

(i) $\lim\limits_{x \to a} \alpha f(x) = \alpha \lim\limits_{x \to a} f(x) = \alpha\, l$ for all $\alpha \in R$

(ii) $\lim\limits_{x \to a} \{f(x) + g(x)\} = \lim\limits_{x \to a} f(x) + \lim\limits_{x \to a} g(x) = l + m$

(iii) $\lim\limits_{x \to a} \{f(x) - g(x)\} = \lim\limits_{x \to a} f(x) - \lim\limits_{x \to a} g(x) = l - m$

(iv) $\lim\limits_{x \to a} \{f(x)\, g(x)\} = \lim\limits_{x \to a} f(x) \lim\limits_{x \to a} g(x) = l \,.\, m$

(v) $\lim\limits_{x \to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim\limits_{x \to a} f(x)}{\lim\limits_{x \to a} g(x)} = \dfrac{l}{m}$ provided $m \neq 0$

(vi) $\lim\limits_{x \to a} \dfrac{1}{f(x)} = \dfrac{1}{\lim\limits_{x \to a} f(x)} = \dfrac{1}{l}$, $\quad l \neq 0$

(vii) $\lim\limits_{x \to a} \{f(x)\}^n = \{\lim\limits_{x \to a} f(x)\}^n = l^n \quad \forall\, n \in N$

(viii) $\lim\limits_{x \to a} |f(x)| = |\lim\limits_{x \to a} f(x)| = |l|$

3. Some standard limits :

(i) $\lim_{x\to 0}\frac{\sin x}{x}=1$

(ii) $\lim_{x\to 0}\frac{\tan x}{x}=1$

(iii) $\lim_{x\to 0}\cos x=1$

(iv) $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{x}=e$

(v) $\lim_{x\to 0}(1+x)^{\frac{1}{x}}=e$

(vi) $\boxed{\lim_{x\to 0}\frac{a^{x}-1}{x}=\log_e a}$

(vii) $\lim_{x\to 0}\frac{e^{x}-1}{x}=1$

(viii) $\lim_{x\to a}\frac{x^{n}-a^{n}}{x-a}=na^{n-1}$

(i) $\lim_{x \to 3} \dfrac{x^2 - 9}{x - 3}$

(ii) $\lim_{x \to 0} \dfrac{\sqrt{1+x} - 1}{x}$

na^{n-1}

$\lim_{x \to 3} \dfrac{x^2 - (3)^2}{x-3} = 2(3)^{2-1}$

$= 6$

$\lim_{x \to 3} \dfrac{\cancel{(x-3)}(x+3)}{\cancel{x-3}}$

$= 3+3 = 6$

$\lim_{x \to 0} \dfrac{\sqrt{1+x} - 1}{x}$

By Rationalising.

$= \lim_{x \to 0} \dfrac{\sqrt{1+x} - 1}{x} \times \dfrac{\sqrt{1+x} + 1}{\sqrt{1+x} + 1}$

$= \lim_{x \to 0} \dfrac{(1+x) - 1}{x(\sqrt{1+x} + 1)} = \lim_{x \to 0} \dfrac{\cancel{x}}{\cancel{x}(\sqrt{1+x} + 1)}$

$= \dfrac{1}{2}$

(i) $\lim_{x \to \infty} \left(\frac{x^2+5}{x^2-5} \right)^{x^2}$ (ii) $\lim_{x \to 0} (1+2x)^{\frac{x+5}{x}}$

$\lim_{x \to \infty} \left(1+\frac{1}{x}\right)^{x} = e$ Reciprocal of each other

$\lim_{x \to 0} (1+x)^{1/x} = e$

$I = \frac{\lim_{x \to \infty} (x^2+5)^{x^2}}{\lim_{x \to \infty} (x^2-5)^{x^2}} = \frac{\lim_{x \to \infty} \cancel{(x^2)^{x^2}} \left(\left(1+\frac{5}{x^2}\right)^{\frac{x^2}{5}}\right)^5}{\lim_{x \to \infty} \cancel{(x^2)^{x^2}} \left(\left(1+\left(\frac{-5}{x^2}\right)\right)^{\frac{x^2}{-5}}\right)^{-5}}$

$= \frac{e^5}{e^{-5}} = e^{10}$

(ii) $\lim_{x \to 0} (1+2x)^{\left(1+\frac{5}{x}\right)}$

$= \lim_{x \to 0} \underbrace{(1+2x)^1} \cdot \underbrace{(1+2x)^{5/x}}$

$= \underbrace{\lim_{x \to 0} (1+2x)} \cdot \lim_{x \to 0} (1+2x)^{5/x}$

$= (1+0) \cdot \lim_{x \to 0} \left[(1+2x)^{\frac{1}{2x}}\right]^{10}$

$= 1 \cdot \lim_{x \to 0} \left[\underbrace{(1+2x)^{\frac{1}{2x}}}\right]^{10}$

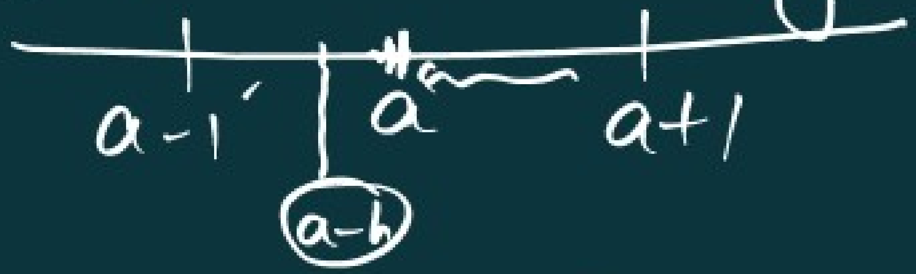
$= \underline{e^{10}}$

Example 8 : (i) If a is an integer, then $\lim_{x \to a^-} [x] = a - 1$ and $\lim_{x \to a^+} [x] = a$

Imp

(ii) If a is not an integer, $\lim_{x \to a} [x] = [a]$, where [x] represents the greatest integer function.

(i) Let a is an integer.



then $\lim_{x \to a^-} [x] = \lim_{h \to 0} [a-h]$

$= a-1$

Now $\lim_{x \to a^+} [x] = \lim_{h \to 0} [a+h]$

$= a$

(ii) Let a is not an integer

$\lim_{x \to a^-} [x] = \lim_{h \to 0} [a-h]$

$= n = [a]$

$\lim_{x \to a^+} [x] = \lim_{h \to 0} [a+h]$

$= n = [a]$

Hence $\lim_{x \to a} [x] = [a]$ if a is non-integer

$[x] = a$; a is nearest inte or smaller to x

$[0.9] = 0$

$[0.9999] = 0$

Theorem : Squeeze Principle (or Sandwich Theorem) :

Statement : If f, g and h are three functions satisfying $f(x) \le h(x) \le g(x)$ in some deleted neighbourhood of 'a' and if $\lim_{x \to a} f(x) = l = \lim_{x \to a} g(x)$, then $\lim_{x \to a} h(x) = l$.

$1 < 2 < 7 < 11$

$2 < 5 < 7$

$1 < 2 < 5 < 7 < 11$

Given is: $f(x) \le h(x) \le g(x)$ —— (1)

$\lim_{x \to a} f(x) = l$:— for some $\epsilon_1 > 0$, $\exists\ \delta_1 > 0$

s.t. $|f(x) - l| < \epsilon_1$ whenever $|x-a| < \delta_1$

$\Rightarrow l - \epsilon_1 < f(x) < l + \epsilon_1$ —— $a - \delta_1 < x < a + \delta_1$ (3)

$\lim_{x \to a} g(x) = l \Rightarrow$ for some $\epsilon_2 > 0$; $\exists\ \delta_2 > 0$.

s.t. $|g(x) - l| < \epsilon_2$ whenever $|x - a| < \delta_2$

$l - \epsilon_2 < g(x) < l + \epsilon_2$ whenever $a - \delta_2 < x < a + \delta_2$ (2)

If $\epsilon_1 < \epsilon_2$

$\delta = \min(\delta_1, \delta_2)$

$l-\epsilon_2 \quad l-\epsilon_1 \quad l \quad l+\epsilon_1 \quad l+\epsilon_2$

$l - \epsilon_2 < l - \epsilon_1 < l + \epsilon_1 < l + \epsilon_2$ (4) where $|x-a| < \delta$

$l - \epsilon_2 < l - \epsilon_1 < f(x) < l + \epsilon_1 < l + \epsilon_2$ when $|x-a| < \delta$
(By using (2))

$\Rightarrow l - \epsilon_2 < f(x) < l + \epsilon_2$

$\Rightarrow l - \epsilon_2 < f(x) \le h(x) \le g(x) < l + \epsilon_2$ whenever $|x-a| < \delta$

$\Rightarrow l - \epsilon_2 < h(x) < l + \epsilon_2$, whenever $|x-a| < \delta$

$\Rightarrow |h(x) - l| < \epsilon_2$, ''

$\Rightarrow \lim_{x \to a} h(x) = l$

Continuity

Definition : Let f be a function defined in the neighbourhood of a. Then the function f is said to be continuous at $x = a$ if for given $\varepsilon > 0$, however small, there exist a real $\delta > 0$ (depending on ε)

such that $\quad |f(x) - f(a)| < \varepsilon \quad$ for $\quad |x - a| < \delta$

$\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = f(a)$ (continuity)

$= (l)$ (limit)

Definition (Discontinuous Function) : A function $f(x)$ is said to be discontinuous at $x = a$ if it is not continuous at that point. A function which is discontinuous even at a single point of an interval is said to discontinuous in that interval.

Continuity $\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x) = f(a)$

Types of discontinuities-

ⓐ Removable discontinuity

$$\underbrace{\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x)}_{} \neq f(a)$$

$$(L.H.L) = (R.H.S) \neq f(a)$$

(b) Discontinuity of Ist kind

$$\lim_{x\to a^-} f(x) \neq \lim_{x\to a^+} f(x)$$

ⓒ Discontinuity of 2nd kind.

Either L.H.L or R.H.L does not exist.

Eg. of Removable discontinuity

$$f(x) = \begin{cases} x^2 & ; x > 0 \\ 2x & ; x < 0 \\ 2 & ; x = 0 \end{cases}$$

L.H.L. $\lim_{x\to 0} 2x = 0$ R.H.S. $\lim_{x\to 0} x^2 = 0$

$f(0) = 2$.

(d) Mixed.

Mix of all above

L.H.L. $\neq f(a)$ R.H.L doesn't exist.

5. Algebra of continuous functions :

Theorem 5.1 : If f and g be two functions continuous at a then

(i) kf is continuous at a

(ii) $f+g$ is continuous at a

(iii) $f-g$ is continuous at a

(iv) fg is continuous at a

(v) $\frac{f}{g}$ is continuous at a if $g(a) \neq 0$

(vi) $\frac{1}{f}$ is continuous at a if $f(a) \neq 0$

Theorem 5.4 : If $f(x)$ is continuous at $x = a$ and $g(x)$ is continuous at $f(a)$, then the composite function $\cancel{f \circ g}$ $g \circ f$ is continuous at $x = a$.

Given $f(x)$ is cts at $x = a$

$\Rightarrow$ for given $\epsilon_1 > 0$, $\exists\ \delta_1 > 0$ s.t.

$|f(x) - f(a)| < \epsilon_1$; $|(x-a)| < \delta_1$ — (1)

$g(x)$ is continuous at $f(a)$

then for given $\epsilon_2 > 0$; $\exists\ \delta_2 > 0$ s.t.

$|g(f(x)) - g(f(a))| < \epsilon_2$ whenever $|f(x) - f(a)| < \delta_2$ — (2)

$a < b \quad b < c$

$a < c$

Let $\epsilon_1 = \delta_2$

then by eqn (2)

$|g(f(x)) - g(f(a))| < \epsilon_2$ whenever $|f(x) - f(a)| < \epsilon_1$ — (3)

By using (1) & (3)

$|g(f(x)) - g(f(a))| < \epsilon_2$ whenever

$\Rightarrow |g\circ f(x) - g\circ f(a)| < \epsilon_2$ if $|x - a| < \delta_1$

. Theorem

If a function f is continuous at a, then it is bounded in some neighbourhood of a.

f is cts at a.

Hint: For any $\epsilon > 0$ $\exists$ $\delta > 0$.

s.t. $\underbrace{|f(x) - f(a)| < \epsilon}$ $\underbrace{\text{whenever } |x-a| < \delta}$.

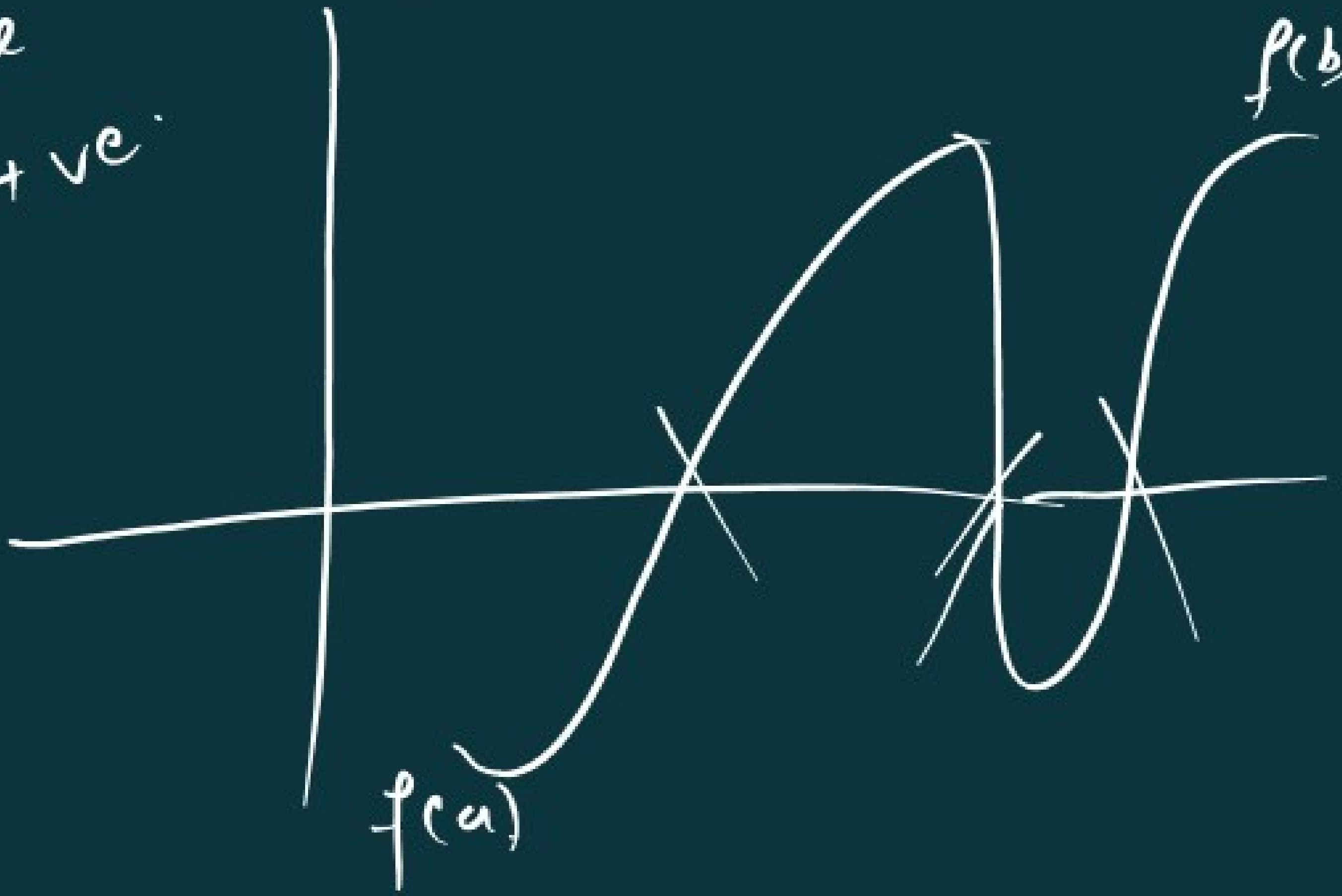
$$-\epsilon < f(x) - f(a) < \epsilon$$

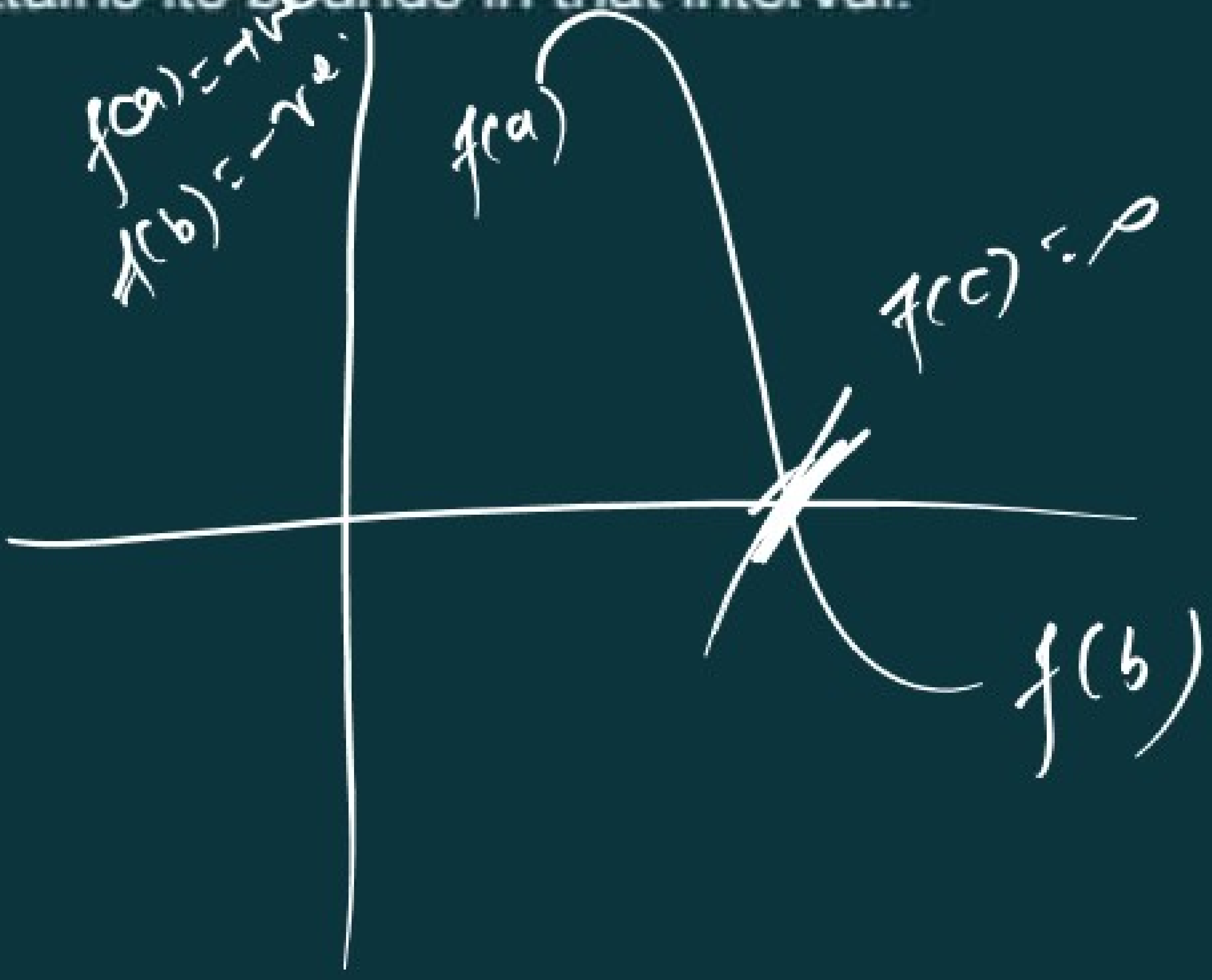
$$\underbrace{f(a) - \epsilon} < f(x) < \underbrace{f(a) + \epsilon}$$

$\Rightarrow$ $f(x)$ is bdd

- Every function defined and continuous on a closed interval is bounded in that interval.
- If f is continuous on a closed and bounded interval [a,b] such that f(a) and f(b) are of opposite signs, then there exists atleast one point c in (a,b) such that f(c)=0.
- Every function defined and continuous on a closed interval attains its bounds in that interval.

$f(a) = -ve$
$f(b) = +ve.$





1.17. Intermediate Value Theorem

If a function f is continuous on a closed interval $[a, b]$ and $f(a) \neq f(b)$, then f assumes every value between $f(a)$ and $f(b)$.

Example 1 : Discuss the continuity of $f(x) = \begin{cases} x \sin \dfrac{1}{x} & \text{when} \quad x \neq 0 \\ 0 & \text{when} \quad x = 0 \end{cases}$ **at** $x = 0$

using $\varepsilon - \delta$ **definition**

[M.D.U. 1997]

Example 2 : Show using $\varepsilon - \delta$ definition that $f(x) = \begin{cases} \sin\frac{1}{x} & , \quad x \neq 0 \\ 0 & , \quad x = 0 \end{cases}$ is discontinuous at $x = 0$ and has discontinuity of second kind.

Example 3 : Show that $f(x)=\begin{cases}\dfrac{|x-a|}{x-a}, & x \neq a \\ k & x=a\end{cases}$ is discontinuous at $x=a$.

Example 5 : Determine the value of k so that the function $f(x) = \begin{cases} 5x - 3k & x \le 1 \\ 3x^2 - kx & x > 1 \end{cases}$ is continuous at $x = 1$.

Example 7 : Examine the continuity of $f(x) = \begin{cases} \dfrac{|x|+x}{2}, & x \le 2 \\ \dfrac{2|x-2|}{x-2} & x > 2 \end{cases}$ at $x = 2$.

Example 8 : Examine the continuity of $f(x)=\begin{cases}\sin^{-1}|x| & \text{if } x\neq 0\\ 0 & \text{if } x=0\end{cases}$ **at** $x=0$

Derivability/ Differentiability

Definition : A function f is said to be derivable at a point a if

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

exists finitely and the limit is called the Derivative of the function at a and is denoted b

$f'(a)$ or $\frac{d}{dx} f(x)$ at $x = a$.

Right hand and Left hand derivatives

Relation between Continuity and Derivability :

Theorem 6.1 : If a function is derivable at a point, then it is necessarily continuous at that point.

Example 1 : Show that the function $f(x) = \begin{cases} (x-a)\cos\dfrac{1}{(x-a)} & , \quad x \neq a \\ 0 & , \quad x = a \end{cases}$ is continuous at a but not derivable at a.

Example 2 : Discuss the continuity and differentiability of $f(x) = \begin{cases} x^2 - 1 & , \quad x \geq 1 \\ 1 - x & , \quad x < 1 \end{cases}$

at $x = 1$. [M.D.U. 1998]

Example 3 : Show that the function $f(x) = \begin{cases} x\left\{\dfrac{e^{\frac{1}{x}}-1}{e^{\frac{1}{x}}+1}\right\}, & x \neq 0 \\ 0, & x = 0 \end{cases}$ is not derivable at $x = 0$.

Example 4 : Show that $f(x) = x\,|\,x\,|$ **is derivable at 0.**

Example 5 : For what values of a and b is the function $f(x) = \begin{cases} x^2 & , \quad x \le c \\ ax + b & , \quad x > c \end{cases}$ differentiable at $x = c$.

Show that $\lim_{x \to 0} \cos\frac{1}{x}$ does not exist.

Let $\langle a_n \rangle = \langle \frac{1}{2n\pi} \rangle$ $\lim_{n\to\infty} a_n\left(\frac{1}{2n\pi}\right) = 0$

$\langle b_n \rangle = \frac{1}{\left(2n\pi + \frac{\pi}{2}\right)}$, $\lim_{n\to\infty} b_n = 0$

Now $f(a_n) = \cos\frac{1}{1/2n\pi}$

$= \cos 2n\pi = 1$

$f(b_n) = \cos\left(2n\pi + \frac{\pi}{2}\right) = \cos\frac{\pi}{2} = 0$

$f(a_n) \neq f(b_n)$

Limit does not exist.

Let $\langle a_n \rangle = \frac{1}{n\pi}$; n is even.

$\langle b_n \rangle = \frac{1}{m\pi}$; m is odd

$f(a_n) = \cos\frac{1}{1/n\pi} = \cos n\pi = (-1)^n = 1$

$f(b_n) = \cos\frac{1}{1/m\pi} = \cos m\pi = (-1)^m = -1$

$f(a_n) \neq f(b_n)$

$\Rightarrow$ Limit doesn't exist.