Statics

Dynamics & Statics

Analytical Geo.

Statics → by Jeevanson's Publication

Dynamics.

Dynamics.

Rectilinear motion, simple harmonic motion, motion in a plane, projectiles; constrained motion; work and energy, conservation of energy;

Kepler's laws, orbits under central forces.

Equilibrium of a system of particles; work and potential energy, friction; common catenary; principle of virtual work; stability of equilibrium, equilibrium of forces in three dimensions.

Statics

REPRESENTATION OF A FORCE BY A STRAIGHT LINE

When a force acts on a body, the following things are necessary to specify it completely :

(i) *its magnitude.*

(ii) *its direction.*

(iii) *its point of application.*

Since a straight line has both magnitude and direction, therefore, a force can be conveniently represented by a straight line equal in length to the magnitude of the force.

Thus a force can completely be represented by a straight line PQ in magnitude and direction as shown in fig. 1, where P is the point of application and $\overrightarrow{PQ}$ represents the line of action and direction from P to Q. $\overrightarrow{QP}$ gives the opposite direction from Q to P.

Fig. 1

$\overrightarrow{PQ}$ $\overrightarrow{QP}$

EQUILIBRIUM OF TWO FORCES

Two forces acting on a body are in equilibrium if they :

(*i*) *are equal in magnitude*

(*ii*) *act along the same line*

(*iii*) *are in opposite direction*.

The converse is also true which states that, if two forces acting at a point on a body are in equilibrium, they must be equal in magnitude and act along the same line in opposite directions.

PRINCIPLE OF INDEPENDENCE OF FORCES

Newton's second law of motion states :

"The rate of change of momentum is directly proportional to the impressed force and takes place in the direction of the force".

The second part of the law reads : "Each force, acting on a body, produces an affect in its own direction, irrespective of the presence of other forces.

This is known as the Principle of Independence of Forces.

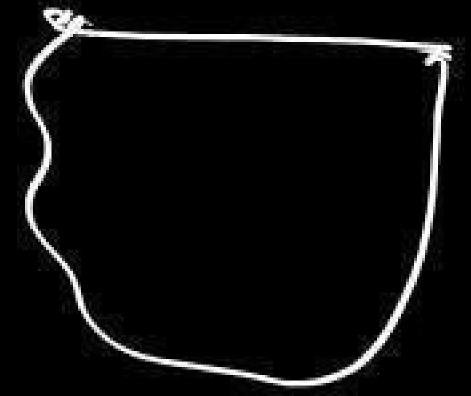
$$\text{Velocity}\,(v) = \frac{ds}{dt}$$

$$\text{Acceleration} = \frac{dv}{dt}$$

$$F = \frac{d(m \times \text{velocity})}{dt}$$

$$= m\,\frac{d(\text{velocity})}{dt}$$

$$= ma.$$



PRINCIPLE OF TRANSMISSIBILITY OF FORCES

Statement. *A force acting at any point of a rigid body may be considered to act at any other point in its line of action provided this latter point is either one of the points of the body or rigidly connected with the body.*

Proof. Let a force F act at a point A of a rigid body along the line AX. Take any point B on the line AX and introduce two equal and opposite forces each equal to F at B, one along BA and the other along BX. The force F acting at A along AB and F at B along BA, being equal, opposite and along the same line of action, neutralise each other and we are left with the force F acting at B along BX.

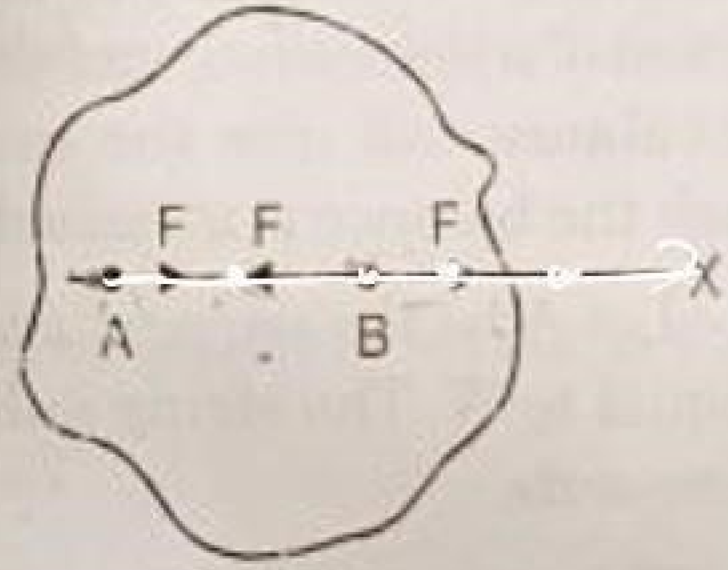


Fig. 2

Hence, the force F at A has been replaced by a force F at any other point B on its line of action.

From the above principle it follows that if two forces F_1 and F_2 act at two different points A and B on a body respectively along the lines intersecting at a point O, then this point may be taken as the point of application of both the forces F_1 and F_2.

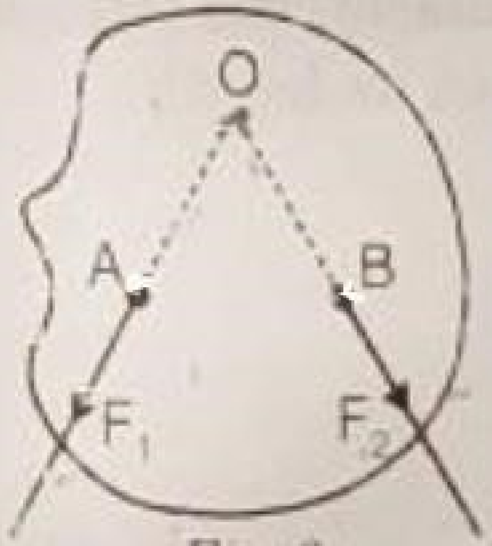


Fig. 3

CLASSIFICATION OF FORCES

There are three different types under which a force may be classified, *viz.* :

(*i*) *Action and reaction*

(*ii*) *Attraction and repulsion*

(*iii*) *Tension and thrust* → outwards.

inward

(i) **Action and Reaction.** Newton's third law of motion states : "*To every action, there is an equal and opposite reaction.*" Whenever two bodies are in contact with each other, each exerts a force on the other at the point of contact. By *action* is meant the force exerted by one body on the other and by *reaction* is meant the force exerted by the second body on the first. These forces are equal in magnitude but opposite in direction and act on different bodies.

The direction of these two forces depends upon the nature of surfaces in contact. For example, when a mass m is placed on the table, the mass presses the table with a force mg downwards, while the table presses the mass with a force R upwards and $R = mg$. Both R and mg act along the common normal in opposite directions.

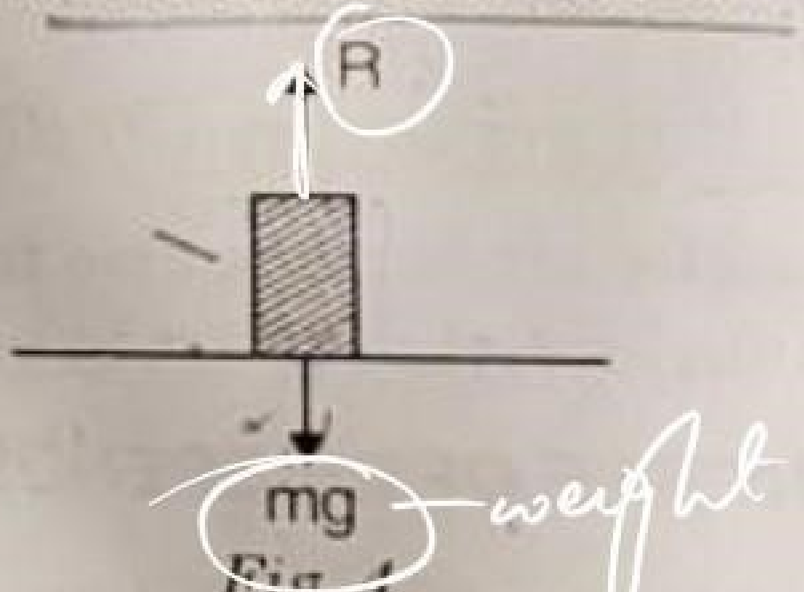


Fig. 1

(*ii*) **Attraction and Repulsion.** Attraction or repulsion is the force exerted by one body on another without any visible or tangible means and without the bodies being necessarily in contact. If the two bodies approach or tend to approach each other, the force is called an *attraction* and the force is called a *repulsion* when the bodies move away or tend to separate. The gravitational pull of the earth towards its centre and the force between two like magnetic poles are the examples of attraction and repulsion respectively.

are the examples of attraction ...

(*iii*) **Tension and Thrust.** When a body is pulled by means of a string or rod, a force is exerted. Such a force applied through a string is called *tension*.

The force exerted by a *light inextensible string* is the same throughout its length. For example, if a weight W is suspended from one end of a light string and the other end is attached to a spring balance, the balance will give the same reading W, whatever be the point at which the balance is attached.

Fig. 5

Let A be the support and the tension T in the string at all points be equal to W. The string pulls the weight *upwards* and the support A *downwards*.

Thus the tension in a string acts in a direction opposite to the body under consideration.

Similarly, the tension in a light string passing over a smooth peg or pulley is the same throughout its length.

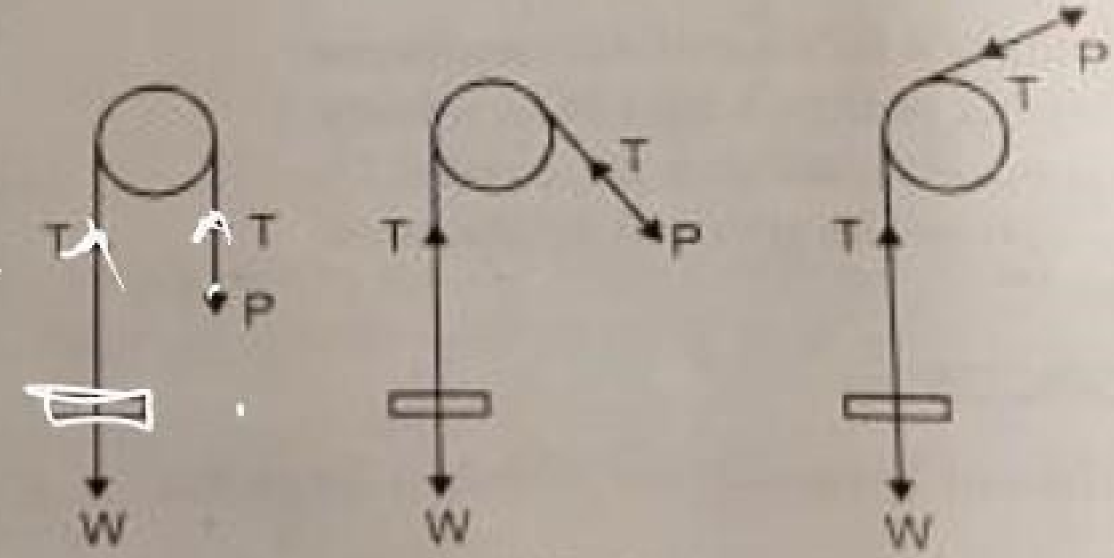


Fig. 6

In fig. 6, a light string supporting a weight W is passing over a smooth pulley; the force P required to support the weight is equal to W, the tension T in the string is the same in all cases and acts along the string in the inward direction and P = T = W.

If, however, the string is knotted at any of its points to other strings or weights, the tensions in the string will not, in general, be the same in different portions of the string.

Thrust. When a light rod is used to exert a *pull*, it behaves in the same way as a light string. However, when the rod is used to exert a push, the forces experienced by the hand and body are *directed towards* the body and the hand and not away from them.

Hence, the thrust in a rod acts always in a direction converging to the body under consideration and is along the rod.

Thus, if a rod AB is used to push the body at A by the application of a force P at B, the thrusts are as shown in fig. 7.

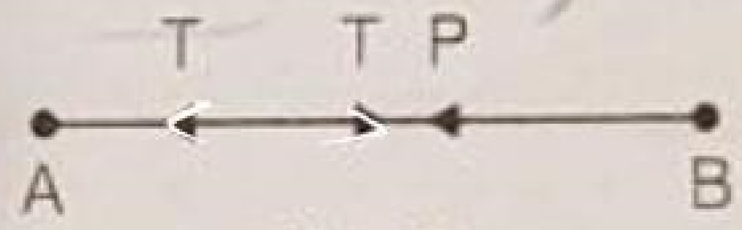


Fig. 7

TENSION IN AN ELASTIC STRING

When one end of an elastic string (or spring) is fixed and a weight is tied to the other end, then the string extends in length. The tension, in this case, is governed by **Hooke's Law**, which states :

"The tension in an elastic string is proportional to the extension of the string beyond its original length."

Thus $$\mathbf{T} = \lambda \cdot \frac{x}{l},$$

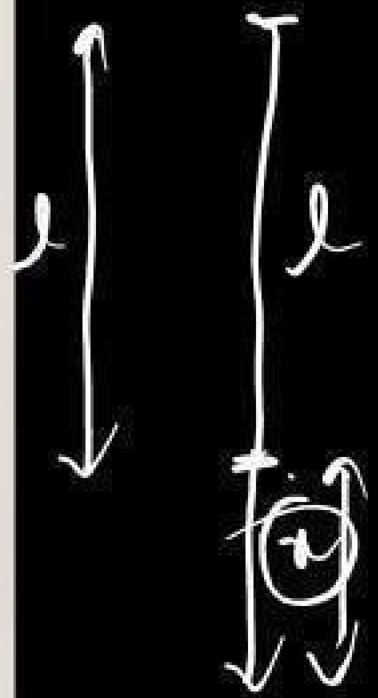
$T \propto \frac{x}{l} = \frac{\text{extension}}{\text{original length}}$

$T = \lambda \frac{x}{l}$

where x = extension in the string beyond its original length.

l = original length of the string

and λ, the constant, is known as the *coefficient of Young's modulus of elasticity* which depends upon the nature of the string.



Cor. If $x = l$, then $\mathrm{T} = \lambda$.

Thus the modulus of elasticity is equal to the tension which extends the string to double its original length.

1.1. RESULTANT AND COMPONENTS

If two or more forces act upon a rigid body and if a single force can be found such that the effect of it upon the body is the same as that of all the forces taken together, then the single force is called the **resultant** *of the forces and the given forces themselves are called the* **components** *of this resultant force.*

Forces are said to be in *equilibrium* if there is no resultant force.

Note. When we say that forces are acting on a particle, it is meant that the forces are acting on a point.

1.2. PARALLELOGRAM LAW OF FORCES

If two forces, acting at a point, be represented in magnitude and direction by the two adjacent sides of a parallelogram through the point of application, their resultant will be completely represented by the diagonal of the parallelogram through that point.

Thus if two forces P and Q acting at a point O be represented in magnitude and direction by the sides OA and OB of the parallelogram OACB, then their resultant R is completely represented by the diagonal OC.

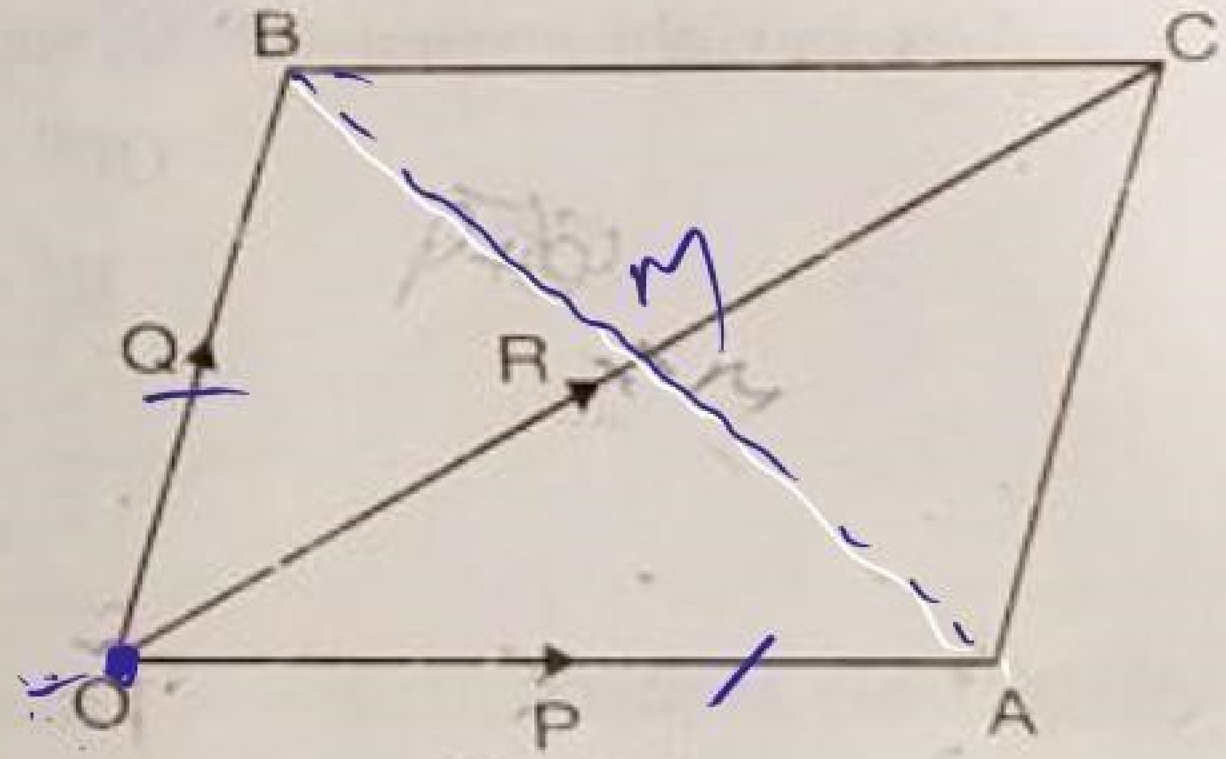


Fig. 1.1

$\overrightarrow{OC} = 2\overrightarrow{OM}$

$= 2\overrightarrow{CM}$

In vector notation, this law may be written as :

$$\boxed{\overrightarrow{OA} + \overrightarrow{OB} = \overrightarrow{OC}}$$

Remark. If the diagonals of the parallelogram meet in M, then M is the middle point of OC. The resultant R is represented by 2OM. In vector notation it can be written as :

$$\overrightarrow{OA} + \overrightarrow{OB} = 2\overrightarrow{OM}\cdot$$

Magnitude and Direction of the Resultants

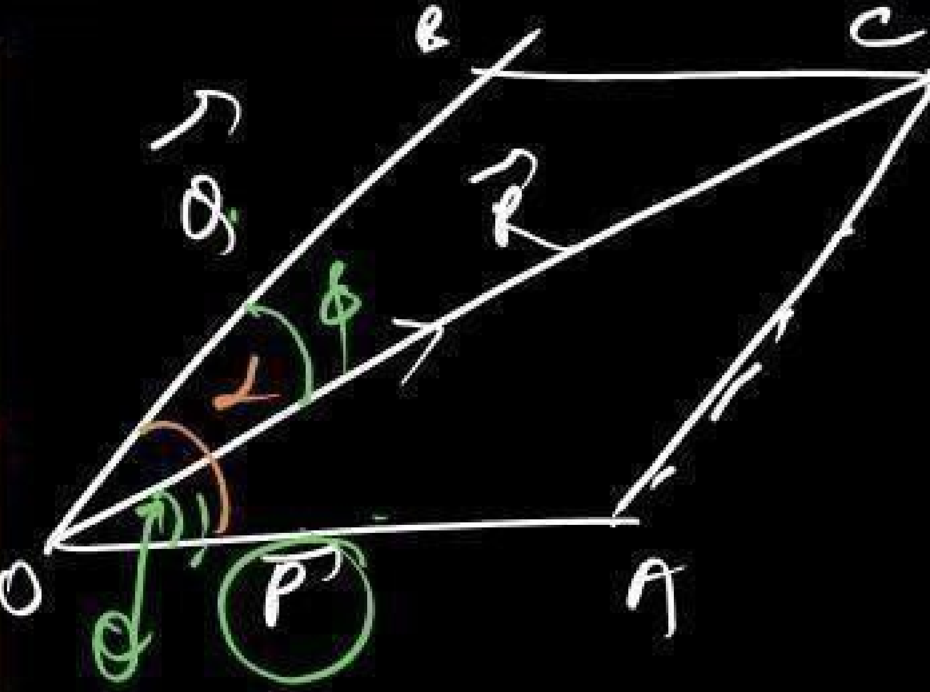
Let the two given forces P and Q acting at an angle α be represented in magnitude and direction by OA and OB respectively. Complete the parallelogram OACB, then the resultant R is represented in magnitude and direction by the diagonal OC.

$$\therefore \quad \boxed{R = \sqrt{P^2 + Q^2 + 2PQ\cos\alpha}}$$

θ -

If θ be the angle which the resultant makes with OA, then

$$\tan\theta = \frac{CD}{OD} = \frac{Q\sin\alpha}{P + Q\cos\alpha} \Rightarrow \boxed{\theta = \tan^{-1}\frac{Q\sin\alpha}{P + Q\cos\alpha}}$$



$\cos\alpha = 1$

$\alpha = 0$

Cor. 1. If ϕ be the angle which the resultant makes with OB, then

$$\tan \phi = \frac{P \sin \alpha}{Q + P \cos \alpha}$$

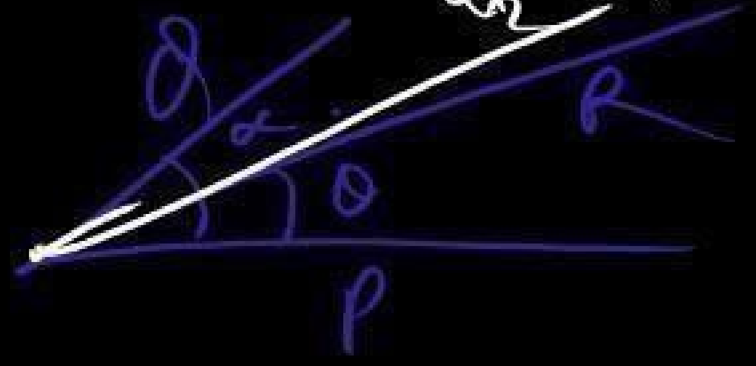
Cor. 2. If the two forces P and Q are perpendicular to one another *i.e.*, if $\alpha = \frac{\pi}{2}$, then

$$R = \sqrt{P^2 + Q^2 + 2PQ \cos \frac{\pi}{2}}$$

$$\therefore \quad R = \sqrt{\mathbf{P^2 + Q^2}} \quad \text{and} \quad \tan \theta = \frac{Q \sin \frac{\pi}{2}}{P + Q \cos \frac{\pi}{2}} = \frac{\mathbf{Q}}{\mathbf{P}}.$$

θ is angle b/w R & P

Let P > Q



$\Rightarrow P + Q\cos\alpha > Q + Q\cos\alpha \Rightarrow \frac{1}{P+Q\cos\alpha} < \frac{1}{Q+Q\cos\alpha}$

$\frac{Q\sin\alpha}{P+Q\cos\alpha} < \frac{Q\sin\alpha}{Q+Q\cos\alpha} \Rightarrow \tan\theta < \frac{\cancel{2}\sin\frac{\alpha}{2}\cancel{\cos\frac{\alpha}{2}}}{\cancel{2}\cos^{\cancel{2}}\frac{\alpha}{2}}$

$\Rightarrow \tan\theta < \tan\frac{\alpha}{2} \Rightarrow \theta < \frac{\alpha}{2}$

Cor 5. Maximum Value of the Resultant

We have $$R^2 = P^2 + Q^2 + 2\,PQ\cos\alpha \qquad \text{...(1)}$$

From (1), R is maximum when $\cos\alpha$ is maximum. But maximum value of $\cos\alpha = 1$, *i.e.*, when $\alpha = 0$

$$\therefore \quad R^2 = P^2 + Q^2 + 2\,PQ = (P + Q)^2$$

$$\therefore \quad R = P + Q.$$

Hence, *the resultant of two forces acting at a point is maximum when they act in the same direction and is equal to their sum.*

Cor. 6. Minimum Value of the Resultant

We have $$R^2 = P^2 + Q^2 + 2PQ\cos\alpha \qquad \text{...(1)}$$

From (1), R is minimum when $\cos\alpha$ is minimum. But minimum value of $\cos\alpha = -1$, *i.e.*, $\alpha = 180°$.

$$\therefore \quad R^2 = P^2 + Q^2 + 2\,PQ\,(-1)$$

$$= (P - Q)^2$$

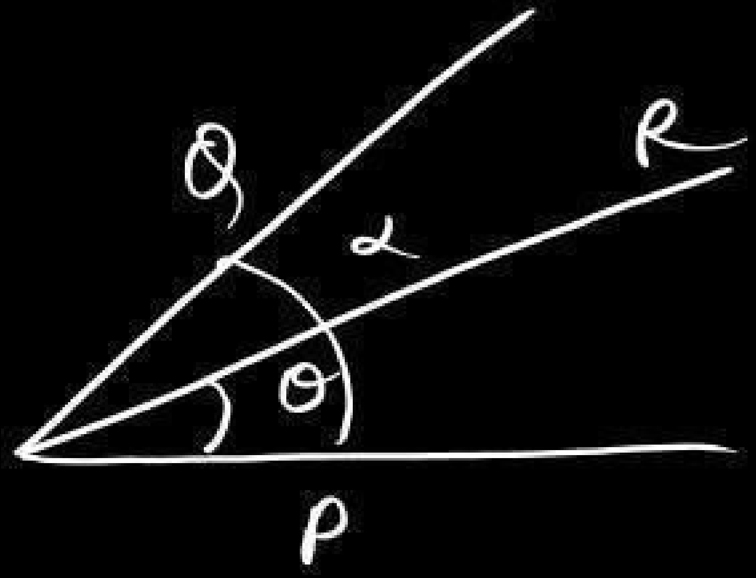
$$\therefore \quad R = P - Q$$

Hence, *the resultant of two forces acting at a point is minimum when they act in opposite directions and is equal to their difference acting in the direction of the greater force.*

Example 2. *Two forces P and 2P act on a particle. If the first be doubled and the second be increased by 10 kg.wt., the direction of the resultant is unaltered. Find the value of P.*

Solution. Let α be the angle between the forces P and 2P and θ be the angle which the resultant makes with P.

$$\tan\theta = \frac{Q\sin\alpha}{P + Q\cos\alpha} \quad \text{—} (1) \qquad P \to 2P, \quad Q \to Q+10$$



$$\tan\theta = \frac{(Q+10)\sin\alpha}{2P + (Q+10)\cos\alpha} \quad \text{—} (2)$$

By (1) & (2)

$$\frac{Q\cancel{\sin\alpha}}{P + Q\cos\alpha} = \frac{(Q+10)\cancel{\sin\alpha}}{2P + (Q+10)\cos\alpha}$$

$Q \to 2P$ Given $\Rightarrow$

$$2P\left(2P + (2P+10)\cos\alpha\right) = (2P+10)\left[P + 2P\cos\alpha\right]$$

$$\Rightarrow 4P^2 + \cancel{4P^2\cos\alpha} + \cancel{20P\cos\alpha} = 2P^2 + 10P + \cancel{20P\cos\alpha} + \cancel{4P^2\cos\alpha}$$

$$\cancel{2}P^2 = \cancel{10}P$$

$$\boxed{P = 5}$$

Example 3. *The greatest and least resultants of two forces are of magnitude P and Q respectively. Show that when they act at an angle θ, their resultant is of magnitude*

$$\sqrt{P^2\cos^2\frac{\theta}{2}+Q^2\sin^2\frac{\theta}{2}}$$

$$\frac{1+\cos 2\theta}{2}=\cos^2\theta \qquad \frac{1-\cos 2\theta}{2}=\sin^2\theta$$

Let f_1 & f_2 are two forces. s.t. $f_1 > f_2$

$$\left.\begin{array}{l} P = f_1+f_2 \\ Q = f_1-f_2 \end{array}\right] \Rightarrow f_1=\frac{P+Q}{2}\ ;\ f_2=\frac{P-Q}{2}$$

$$\text{Resultant} = \sqrt{f_1^2+f_2^2+2f_1f_2\cos\theta}$$

$$= \sqrt{\left(\frac{P+Q}{2}\right)^2+\left(\frac{P-Q}{2}\right)^2+2\frac{(P+Q)(P-Q)}{4}\cos\theta}$$

$$= \sqrt{\frac{P^2+Q^2+2PQ}{4}+\frac{P^2+Q^2-2PQ}{4}+\frac{2(P^2-Q^2)}{4}\cos\theta}$$

$$= \sqrt{\frac{2P^2+2Q^2+2P^2\cos\theta-2Q^2\cos\theta}{4}}$$

$$= \sqrt{\frac{2P^2(1+\cos\theta)}{4}+\frac{2Q^2(1-\cos\theta)}{4}}$$

$$= \sqrt{P^2\frac{(1+\cos\theta)}{2}+Q^2\frac{(1-\cos\theta)}{2}}$$

$$= \sqrt{P^2\cos^2\frac{\theta}{2}+Q^2\sin^2\frac{\theta}{2}}$$

Example 5. *The resultant of two forces P and Q is of magnitude Q. Show that if the force Q be doubled, P remaining the same, the new resultant will be at right angle to P and its magnitude will be* $\sqrt{4Q^2 - P^2}$.

two forces → P & Q ⇒ $R^2 = P^2 + Q^2 + 2PQ\cos\alpha$; where α is angle b/w P & Q

$R = Q$

$Q \to 2Q$

But $R = Q$

$\therefore \cancel{Q^2} = P^2 + \cancel{Q^2} + 2PQ\cos\alpha$

$\Rightarrow P^2 + 2PQ\cos\alpha = 0$

$\Rightarrow P(P + 2Q\cos\alpha) = 0$

$\Rightarrow \boxed{P + 2Q\cos\alpha = 0}$ —— (1)

$\cos\alpha = \frac{-P}{2Q}$

According to Given Quesⁿ :-

Let R' be new resultant & θ_1 is the angle made by R' with P then →

$\tan\theta_1 = \dfrac{2Q\sin\alpha}{P + 2Q\cos\alpha}$ —— (2)

By (1) & (2) $\tan\theta_1 = \infty \Rightarrow \theta_1 = 90^\circ$

To find out resultant

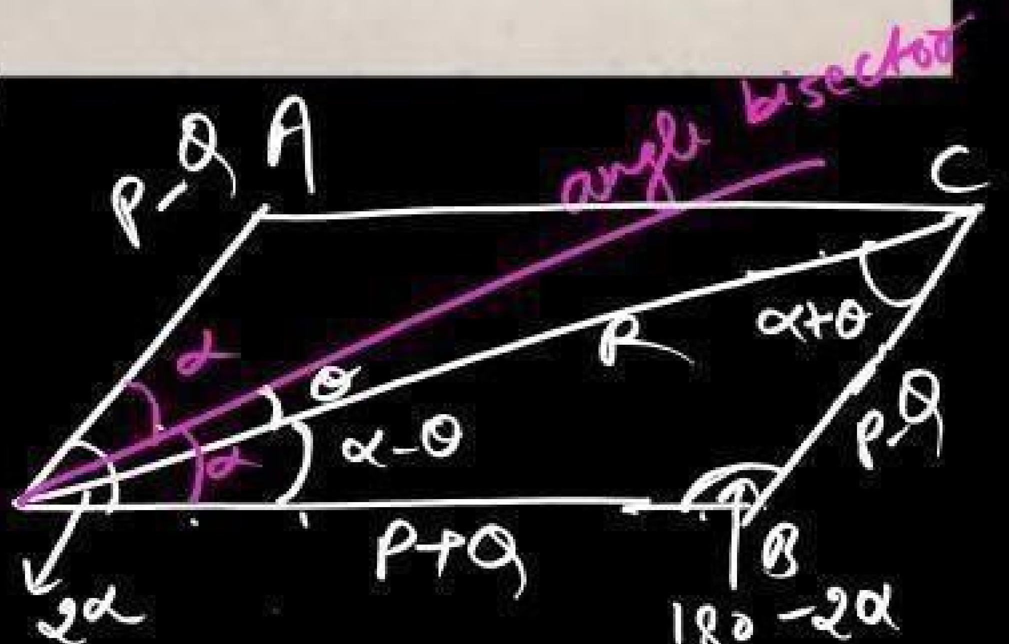
$R' = \sqrt{P^2 + (2Q)^2 + 2P(2Q)\cdot\cos\alpha}$

$= \sqrt{P^2 + 4Q^2 + 4PQ\cos\alpha}$

$= \sqrt{P^2 + 4Q^2 + \cancel{4}^{2}P\cancel{Q}\left(-\frac{P}{\cancel{2Q}}\right)} = \sqrt{4Q^2 - P^2}$ Ans

Example 6. *Two forces P + Q and P − Q make an angle 2α with one another and their resultant makes an angle θ with the bisector of the angle between them. Show that*

$$P \tan\theta = Q \tan\alpha$$



Now

$$\boxed{\tan(\alpha-\theta) = \frac{(P-Q)\sin 2\alpha}{(P+Q)+(P-Q)\cos 2\alpha}}$$

Applying ~~sine~~ sine formula on △OBC

$$\frac{\sin\angle COB}{BC} = \frac{\sin\angle B}{OC} = \frac{\sin\angle OCB}{OB}$$

✓ $$\frac{\sin(\alpha-\theta)}{P-Q} = \frac{\sin(180-2\alpha)}{R} = \frac{\sin(\alpha+\theta)}{P+Q}$$

comparing ① & ③ fraction → $\frac{\sin(\alpha-\theta)}{P-Q} = \frac{\sin(\alpha+\theta)}{P+Q}$

$$(P+Q)(\sin\alpha\cos\theta - \cos\alpha\sin\theta) = (P-Q)(\sin\alpha\cos\theta + \cos\alpha\sin\theta)$$

$$-P\cos\alpha\sin\theta + Q\sin\alpha\cos\theta = P\cos\alpha\sin\theta - Q\sin\alpha\cos\theta$$

$$\Rightarrow \cancel{2}P\cos\alpha\sin\theta = \cancel{2}Q\sin\alpha\cos\theta$$

$$\Rightarrow P\frac{\sin\theta}{\cos\theta} = Q\frac{\sin\alpha}{\cos\alpha}$$

$$\Rightarrow P\tan\theta = Q\tan\alpha$$

(Sum of two adjacent angle of ||al. is 180)

$$\angle OCB = 180 - (180-2\alpha) - (\alpha-\theta)$$

$$\cancel{180} - \cancel{180} + 2\alpha - \alpha + \theta$$

$$= \alpha + \theta$$

Example 7. *If the greatest possible resultant of two forces P and Q is n times the least, show that the angle between them when their resultant is half of their sum is* $\cos^{-1}\left(-\frac{n^2+2}{2(n^2-1)}\right)$.

Given forces are P & Q

Greatest rest^n = $P+Q$

Least rest^n = $P-Q$

A.T.Q $\Rightarrow P+Q = n(P-Q)$

$P+Q = nP - nQ$

$P(1-n) = -(1+n)Q$

$\boxed{P = \frac{(n+1)}{n-1}Q}$

Also $R = \frac{P+Q}{2}$ — (1)

Now $R^2 = P^2+Q^2+2PQ\cos\alpha$; α is angle b/w P & Q — (2)

By (1) & (2)

$\left(\frac{P+Q}{2}\right)^2 = P^2+Q^2+2PQ\cos\alpha$

$\Rightarrow P^2+Q^2+2PQ = 4P^2+4Q^2+8PQ\cos\alpha$

$\Rightarrow 3P^2+3Q^2+8PQ\cos\alpha - 2PQ = 0$

$\Rightarrow 3\left(\frac{n+1}{n-1}\right)^2Q^2+3Q^2+8\left(\frac{n+1}{n-1}\right)Q^2\cos\alpha - \frac{2(n+1)}{(n-1)}Q^2 = 0$

$\Rightarrow \frac{3(n+1)^2}{(n-1)^2}+3+8\frac{(n+1)}{n-1}\cos\alpha - \frac{2(n+1)}{n-1} = 0$

$\Rightarrow \cos\alpha = \frac{n-1}{8(n+1)}\left[2\frac{(n+1)}{(n-1)} - 3\frac{(n+1)^2}{(n-1)^2} - 3\right]$

$= \frac{\cancel{n-1}}{8(n+1)}\left[\frac{2(n^2-1) - 3(n^2+1+2n) - 3(n^2+1-2n)}{(n-1)^2}\right]$

$\cos\alpha = \frac{-4n^2-8}{8(n^2-1)} = \boxed{\cos^{-1}\left(-(n^2+2)/2(n^2-1)\right) = \alpha}$

Example 9. *If forces P and Q acting at an angle θ be interchanged in position, show that their resultant turns through an angle φ such that,*

$$\tan\frac{\phi}{2} = \frac{P-Q}{P+Q}\tan\frac{\theta}{2}$$

Let $P > Q$ & γ be the angle b/w P & resultant

In $\triangle OAB$

$$\frac{\sin\left(\frac{\theta-\phi}{2}\right)}{AB = Q} = \frac{\sin\left(\frac{\theta+\phi}{2}\right)}{OA = P} = \underbrace{\frac{\sin(180-\theta)}{OB = R}}$$

$$P\sin\left(\frac{\theta-\phi}{2}\right) = Q\sin\left(\frac{\theta+\phi}{2}\right)$$

$$P\left(\sin\frac{\theta}{2}\cos\frac{\phi}{2} - \cos\frac{\theta}{2}\sin\frac{\phi}{2}\right) = Q\sin\frac{\theta}{2}\cos\frac{\phi}{2} + Q\cos\frac{\theta}{2}\sin\frac{\phi}{2}$$

$$(P-Q)\sin\frac{\theta}{2}\cos\frac{\phi}{2} = (P+Q)\cos\theta_{/2}\sin\frac{\phi}{2}$$

$$\frac{P-Q}{P+Q}\tan\theta_{/2} = \tan\phi_{/2}$$

$\angle DOE = \theta - \gamma = \theta - \left(\frac{\theta-\phi}{2}\right) = \frac{\theta+\phi}{2}$

$\angle OED = 180 - \theta$

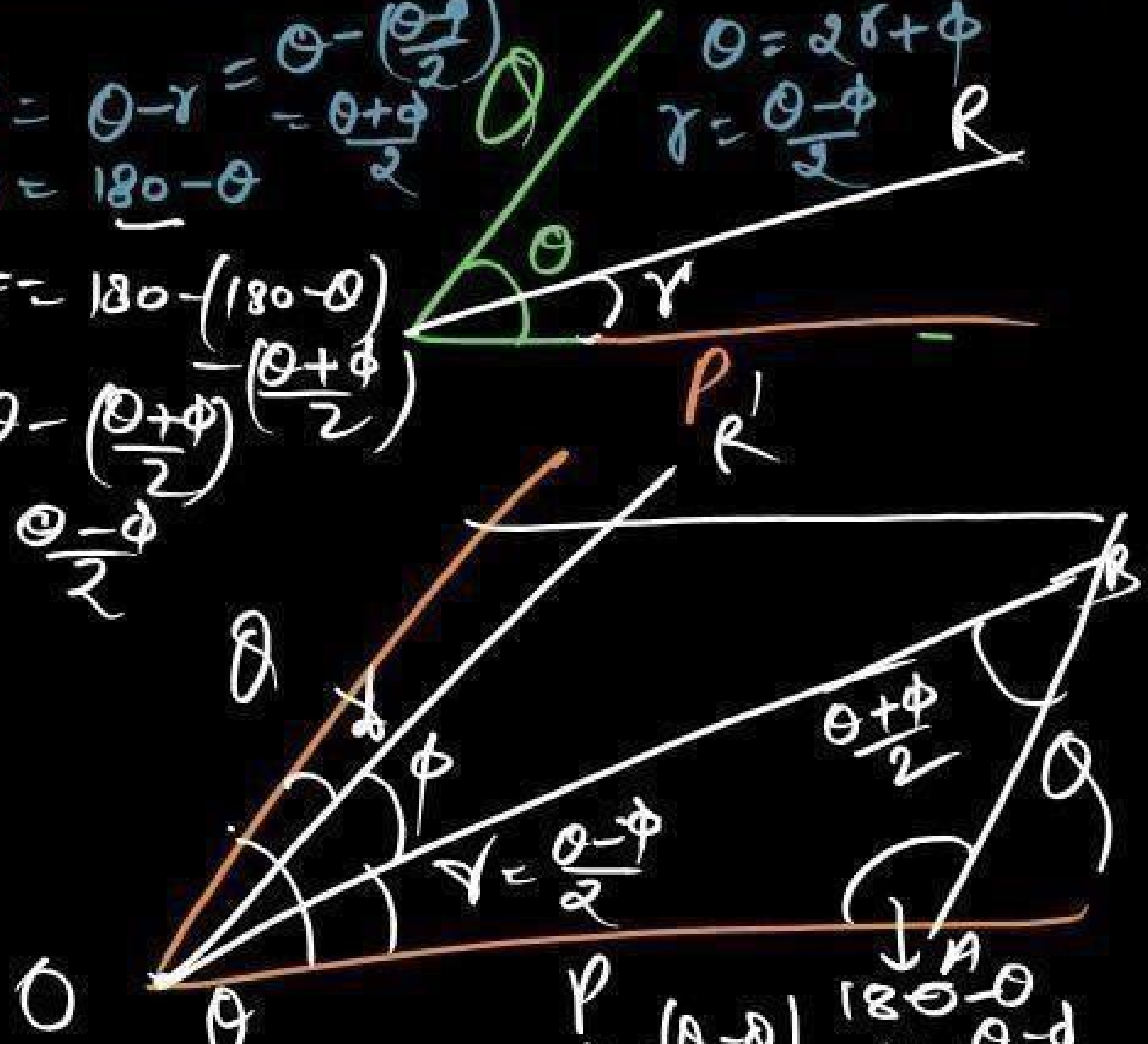
$\theta = 2\gamma + \phi$

$\gamma = \frac{\theta-\phi}{2}$

$\angle ODE = 180 - (180-\theta) - \left(\frac{\theta+\phi}{2}\right)$

$= \theta - \left(\frac{\theta+\phi}{2}\right)$

$= \frac{\theta-\phi}{2}$



$180 - (180-\theta) - \left(\frac{\theta-\phi}{2}\right) = \theta - \frac{\theta-\phi}{2} = \frac{\phi+\theta}{2}$

1.4. RESOLUTION OF A GIVEN FORCE IN TWO GIVEN DIRECTIONS

Resolution of forces is the converse of the *composition of forces*. When a force is given, we are to find the *component forces* in given directions. Given a resultant force in magnitude and direction, we can resolve it into two components in an infinite number of ways; since on a given line as diagonal an *infinite* number of parallelograms can be constructed.

Applying sine formula on $\triangle OAC$.

$$\frac{P}{\sin\beta} = \frac{Q}{\sin\alpha} = \frac{F}{\sin(180-(\alpha+\beta))}$$

$$\Rightarrow P = \frac{F\sin\beta}{\sin(\alpha+\beta)}, \quad Q = \frac{F\sin\alpha}{\sin(\alpha+\beta)}$$

1.5. COMPONENTS OF A GIVEN FORCE IN TWO GIVEN DIRECTIONS

Let F be the given force represented in magnitude and direction by OC and let OX, OY be the given directions making angles α, β respectively with OC along which components are to be found.

Through C, draw CA and CB parallel to OY and OX respectively to meet OX and OY in A and B. Then OACB is a parallelogram and OA, OB are the required components of the force F in magnitude and direction.

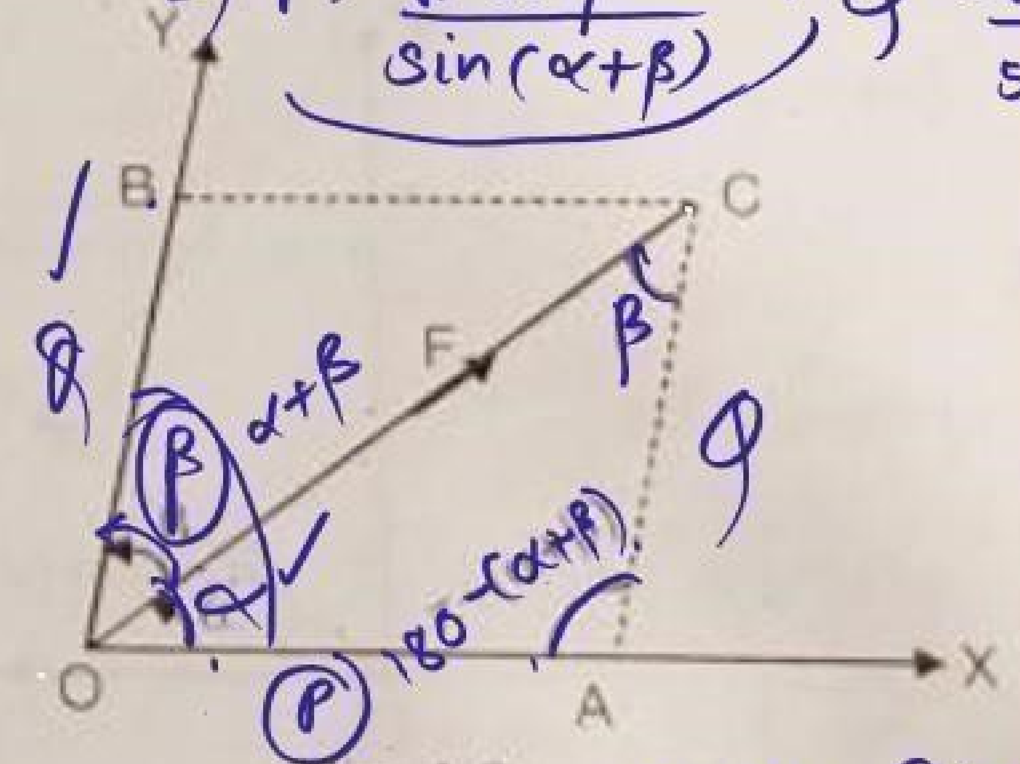


Fig. 1.8

$\alpha+\beta = 90 \Rightarrow \alpha = 90-\beta.$

Now, AC is equal to and parallel to OB.

Thus, both represent the same force in magnitude and direction. $Q = F\sin(90-\beta) = F\cos\beta$

Now, in Δ OAC, $\angle AOC = \alpha$, $\angle OCA = \angle COB = \beta$

and $\angle OAC = \pi - (\alpha + \beta)$

From ΔOAC, by using sine formula, we have

$$\frac{OA}{\sin \angle OCA} = \frac{AC}{\sin \angle AOC} = \frac{OC}{\sin \angle CAO}$$

or $$\frac{OA}{\sin \beta} = \frac{OB}{\sin \alpha} = \frac{OC}{\sin [\pi - (\alpha + \beta)]}$$

or $$\frac{OA}{\sin \beta} = \frac{OB}{\sin \alpha} = \frac{F}{\sin (\alpha + \beta)}$$

Hence, $$OA = \frac{F \sin \beta}{\sin (\alpha + \beta)} \quad \text{and} \quad OB = \frac{F \sin \alpha}{\sin (\alpha + \beta)}$$

In words : $\boxed{\textbf{\textit{Components in any direction}} = \dfrac{\textbf{\textit{F}} \times \textbf{\textit{sine (other angle)}}}{\textbf{\textit{sine (sum of angles)}}}}$

Note. By other angle we mean the angle which the other direction makes with the given force.

1.6. RESOLVED PARTS OF A GIVEN FORCE

Definition. *If a force be resolved into two components, which are at right angles to each other, then these components are called the resolved parts of the force.*

$\therefore$ ***Resolved part of a force in any direction***

= Force × cosine of the angle which the force makes with that direction.

Cor 1. The resolved part of a force F in its own direction

$$= F \cos 0° = F \qquad [\because \alpha = 0°]$$

Cor 2. The resolved part of a force F in a direction perpendicular to it

$$= F \cos 90° = 0 \qquad [\because \alpha = 90°]$$

Hence a force cannot produce any effect in a direction perpendicular to its line of action.

Cor 3. The resolved part of a given force F in a given direction represents the *whole effect* of the force in that direction.

Example 2. *The resultant of two forces P and Q is R. The resolved part of R in the direction of P is of magnitude Q. Show that the angle between the forces is*

$$2\sin^{-1}\sqrt{\frac{P}{2Q}}.$$

$P + Q\cos\alpha = Q$

$P = Q(1-\cos\alpha)$

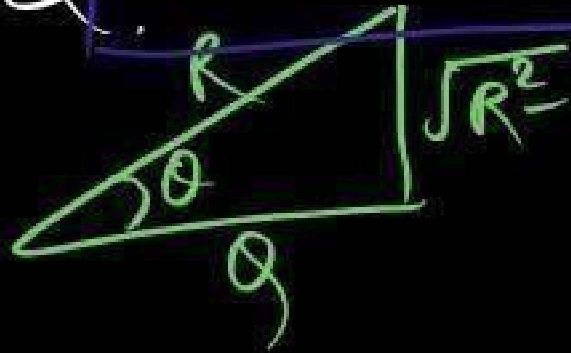
$P = 2Q\sin^2\frac{\alpha}{2} \Rightarrow \sin\frac{\alpha}{2} = \sqrt{\frac{P}{2Q}}$

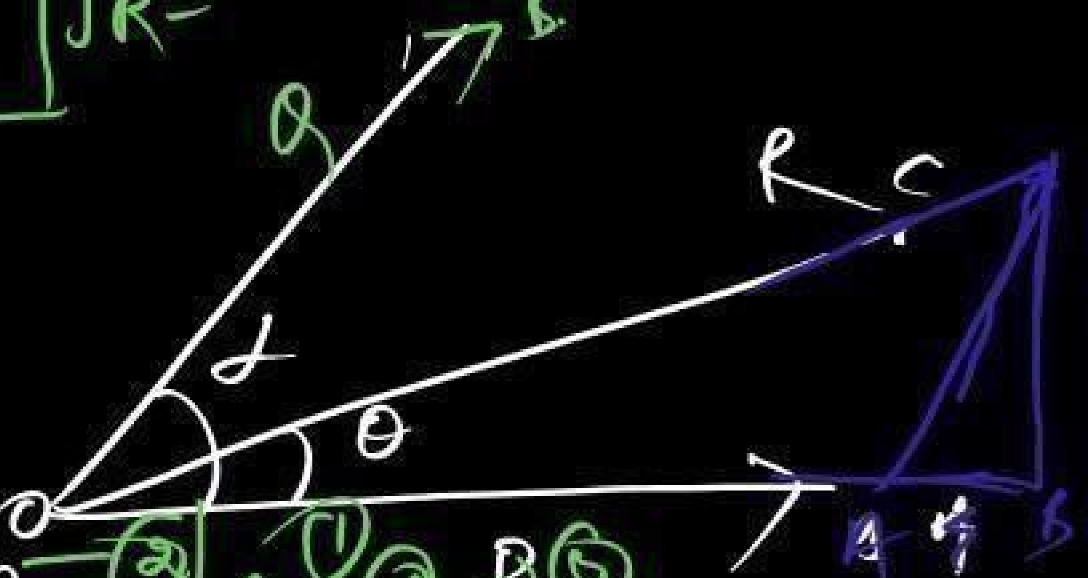
$\alpha = 2\sin^{-1}\sqrt{P/2Q}$

Let angle b/w two forces P & Q is α.

& Given R is the resultant.

$R^2 = P^2 + Q^2 + 2PQ\cos\alpha$ — (1)





Resolved part of R in the direction of P = $\underbrace{R\cos\theta}$

A.T.Q. $R\cos\theta = Q$

$\cos\theta = \frac{Q}{R}$

Also $\tan\theta = \frac{Q\sin\alpha}{P+Q\cos\alpha}$ — (2)

$\tan\theta = \frac{\sqrt{R^2-Q^2}}{Q}$ — (3)

By (1), (2) & (3)

$\frac{Q\sin\alpha}{P+Q\cos\alpha} = \frac{\sqrt{P^2+2PQ\cos\alpha}}{Q}$

$\frac{Q^2\sin^2\alpha}{P^2+Q^2\cos^2\alpha+2PQ\cos\alpha} + 1 = \frac{P^2+2PQ\cos\alpha}{Q^2} + 1$

$\Rightarrow (Q^2\sin^2\alpha + P^2 + \cancel{Q^2\cos^2\alpha} + 2PQ\cos\alpha)/(P+Q\cos\alpha)^2 = \frac{P^2+Q^2+2PQ\cos\alpha}{Q^2}$

Example 3. *Two forces P and Q, acting on a particle are inclined at the angle θ. If the sum of their resolved parts in a certain direction be X and that along a perpendicular direction be Y, prove that*

$$\theta = \cos^{-1}\frac{X^2+Y^2-P^2-Q^2}{2PQ}.$$

Let α is the angle made by P along OA

Resolved part of P along $OA = P\cos\alpha$

Resolved part of Q along $OA = Q\cos(\alpha+\theta)$

Resolved part of P along $OB = P\cos(90-\alpha) = P\sin\alpha$

Resolved part of Q along $OB = Q\cos(90-(\alpha+\theta)) = Q\sin(\alpha+\theta)$

A.T.Q:- $X = P\cos\alpha + Q\cos(\alpha+\theta)$ — (1)

$Y = P\sin\alpha + Q\sin(\alpha+\theta)$ — (2)

$(1)^2+(2)^2 = X^2+Y^2 = P^2\cos^2\alpha + Q^2\cos^2(\alpha+\theta) + 2PQ\cos\alpha\cos(\alpha+\theta) + P^2\sin^2\alpha + Q^2\sin^2(\alpha+\theta) + 2PQ\sin\alpha\sin(\alpha+\theta)$

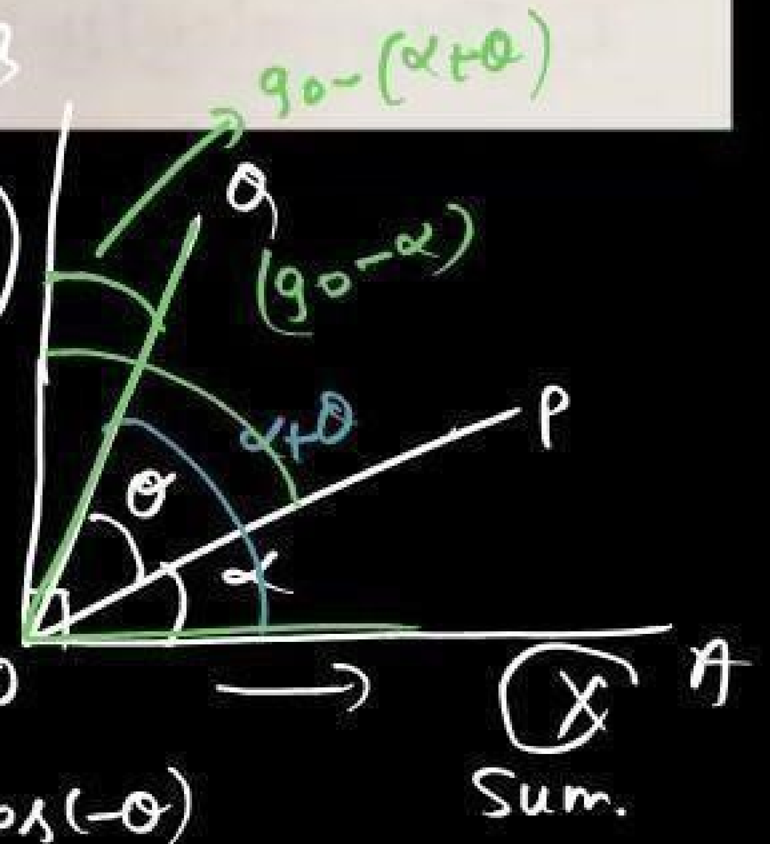
$\Rightarrow X^2+Y^2 = P^2+Q^2+2PQ\,(\cos\alpha\cos(\alpha+\theta) + \sin\alpha\sin(\alpha+\theta))$

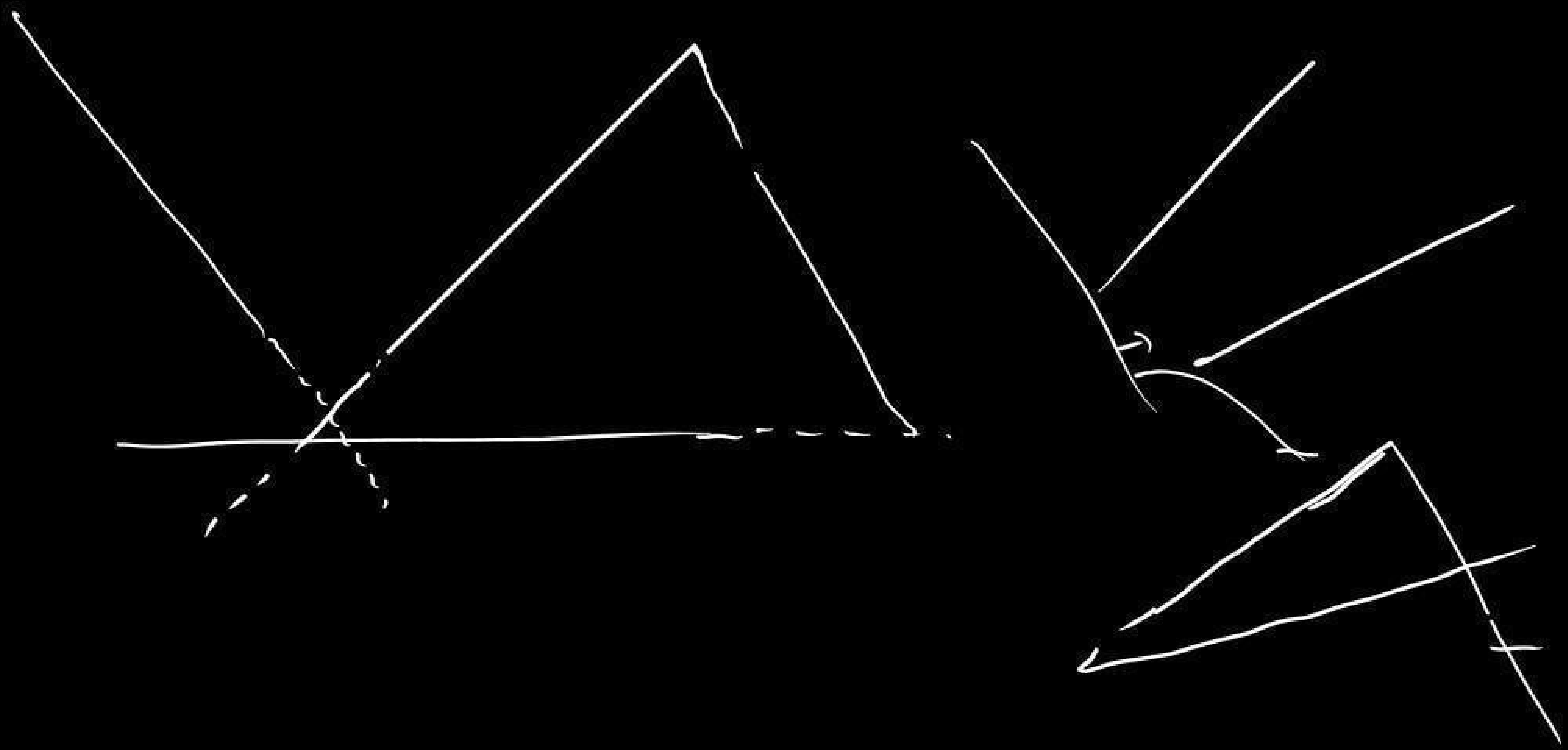
$\Rightarrow X^2+Y^2 = P^2+Q^2+2PQ\,(\cos(\alpha-\alpha-\theta))$

$\Rightarrow X^2+Y^2 = P^2+Q^2+2PQ\cos(-\theta)$

$\Rightarrow \dfrac{X^2+Y^2-P^2-Q^2}{2PQ} = \cos\theta.$

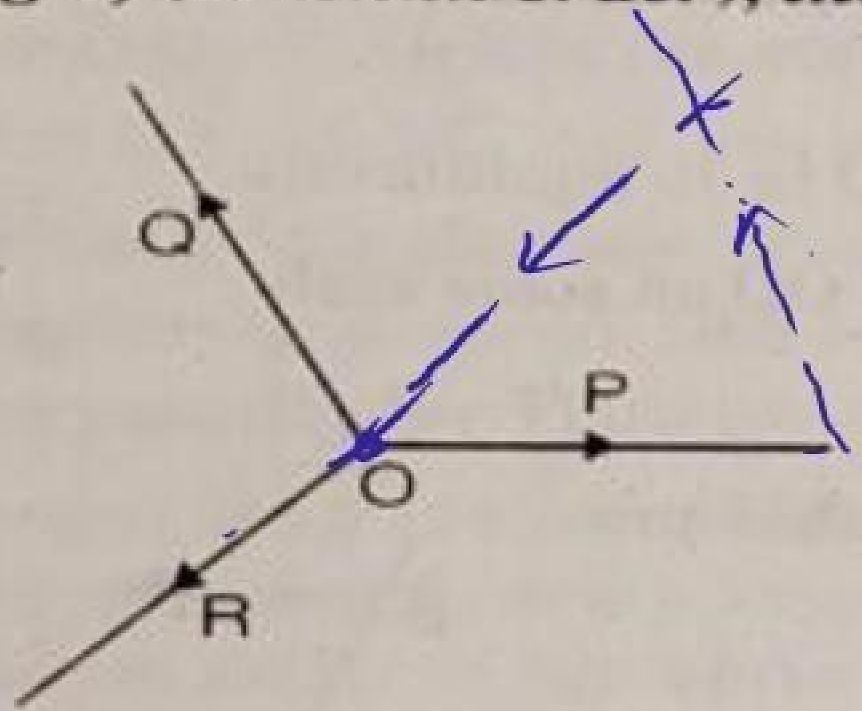
$\Rightarrow \theta = \cos^{-1}\left(\dfrac{X^2+Y^2-P^2-Q^2}{2PQ}\right)$

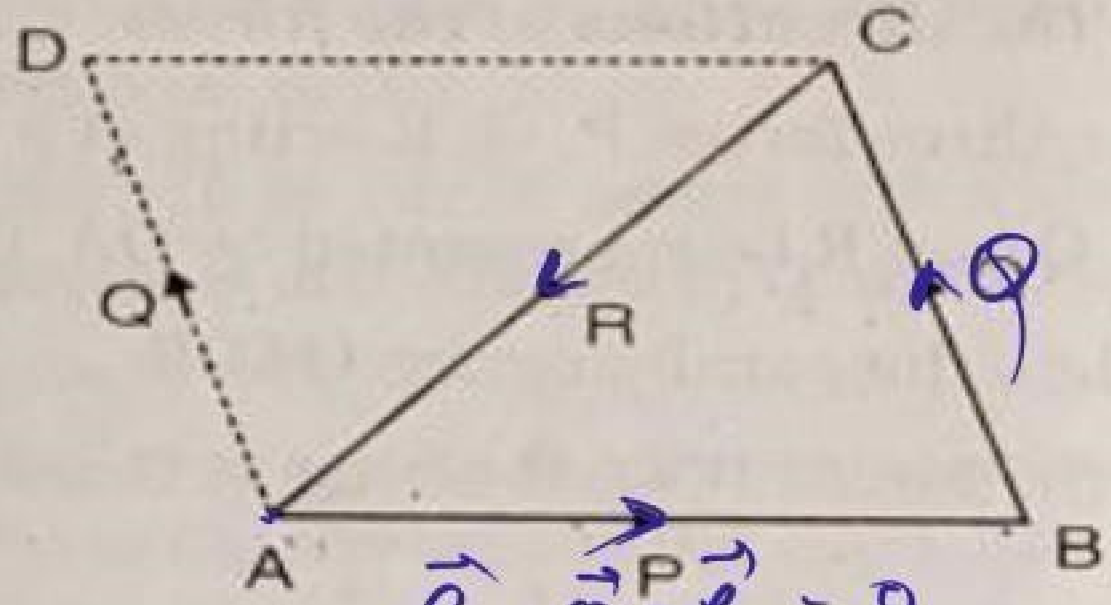




1.7. TRIANGLE LAW OF FORCES

If three forces, acting at a point, be represented in magnitude and direction by the sides of a triangle, (taken in order), then the forces are in equilibrium.





$\vec{P} + \vec{Q} + \vec{R} = 0$

$\Rightarrow \vec{P} + \vec{Q} = -\vec{R}$

$\vec{AB} + \vec{BC} = \vec{AC}$

Hence the triangle law of forces can also be stated as :

If two forces, acting at a point, be represented in magnitude and direction by two sides of a triangle taken in order, their resultant is represented in magnitude and direction by the third side taken in the opposite order.

Note. It should be noted that the forces must act at a point and they are represented by the sides of the triangle in magnitude and direction only and not in the line of action.

1.8. CONVERSE OF THE TRIANGLE LAW OF FORCES

If three forces acting at a point be in equilibrium, they can be represented by the sides (taken in order) of any triangle which is drawn so as to have its sides respectively parallel to the directions of the forces.

1.10. $\lambda - \mu$ THEOREM

The resultant of two forces acting at a point O in direction OA and OB and represented in magnitude by λ.OA and μ.OB is represented by $(\lambda + \mu)$.OC, where C is a point in AB such that λ.CA = μ.CB. [K.U. 2000]

Proof. Let the two given forces λ.OA and μ.OB act at the point O in the direction OA and OB respectively.

Take a point C on AB such that λ.CA = μ.CB.

From the ΔOAC, by the triangle law of forces, using vector notation, we have

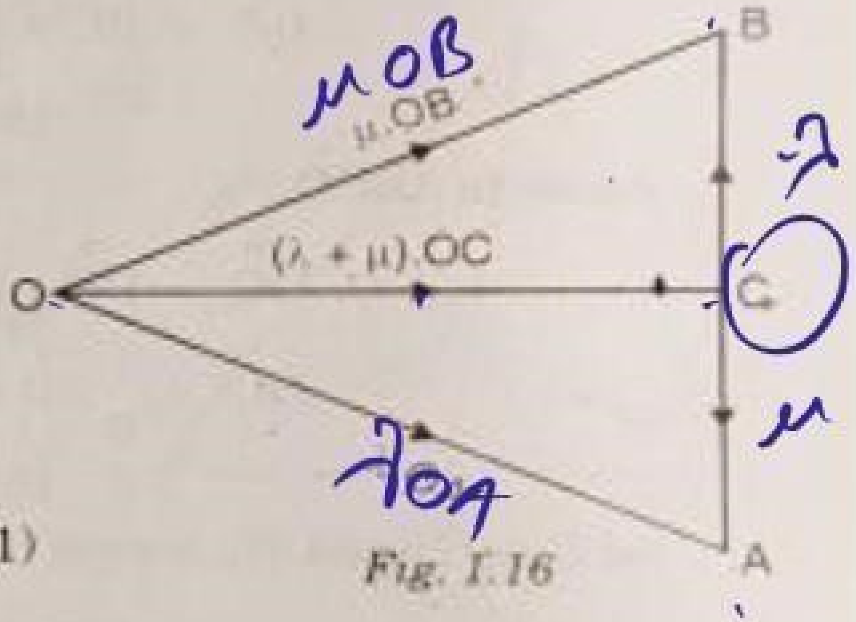


Fig. 1.16

$$\overrightarrow{OA} = \overrightarrow{OC} + \overrightarrow{CA}$$

$$\therefore \quad \lambda.\overrightarrow{OA} = \lambda.\overrightarrow{OC} + \lambda.\overrightarrow{CA} \qquad ...(1)$$

Similarly, from ΔOCB,

$$\therefore \quad \mu.\overrightarrow{OB} = \mu.\overrightarrow{OC} + \mu.\overrightarrow{CB} \qquad ...(2)$$

Adding (1) and (2), we have

$$\lambda.\overrightarrow{OA} + \mu.\overrightarrow{OB} = (\lambda + \mu).\overrightarrow{OC} + \lambda.\overrightarrow{CA} + \mu.\overrightarrow{CB}$$

$$= (\lambda + \mu).\overrightarrow{OC} + \lambda.\overrightarrow{CA} - \mu.\overrightarrow{BC}$$

$$= (\lambda + \mu).\overrightarrow{OC} \qquad [\because \lambda.\overrightarrow{CA} = \mu.\overrightarrow{BC}]$$

Hence the result.

$$\frac{CA}{CB} = \frac{\mu}{\lambda}$$

$$\Rightarrow \lambda CA = \mu CB.$$

Cor. 1. *The resultant of two forces $\overrightarrow{OA}$ and $\overrightarrow{OB}$ is a force $2.\overrightarrow{OC}$ where C is the middle point of AB.*

Example 1. *If D and E are the middle points of the sides AB and AC of a triangle ABC, prove that the resultant of the forces represented by BE and DC is represented in magnitude and direction by* $\frac{3}{2}$*BC.*

In $\triangle EBC$

$\vec{BE} = \vec{BC} + \vec{CE}$ — (1)

In $\triangle DBC$

$\vec{DC} = \vec{DB} + \vec{BC}$ — (2)

(1) + (2) ⇒

$\vec{BE} + \vec{DC} = \vec{BC} + \vec{CE} + \vec{DB} + \vec{BC}$

$= 2\vec{BC} + \vec{CE} + \vec{DB}$

$= 2\vec{BC} + \frac{1}{2}\vec{CA} + \frac{1}{2}\vec{AB}$

$= 2\vec{BC} + \frac{1}{2}(\vec{CA} + \vec{AB})$ — (3)

In $\triangle ABC$

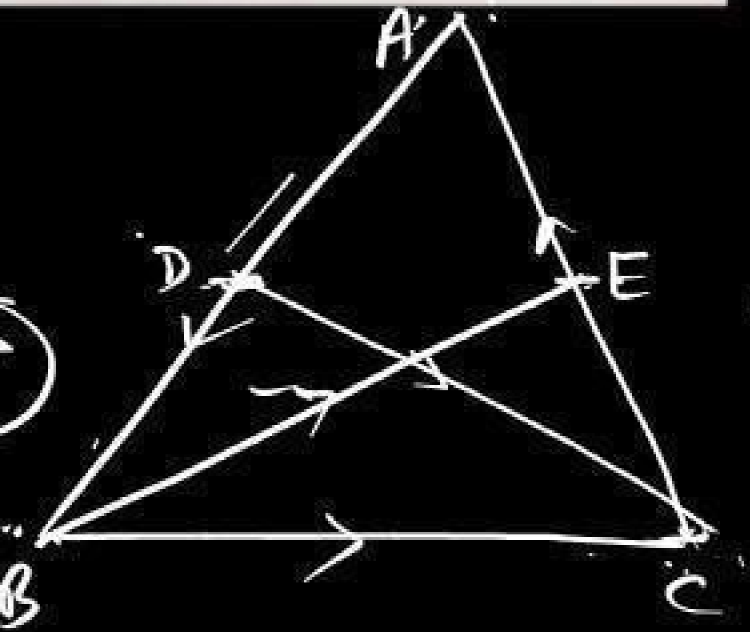
$\vec{CA} + \vec{AB} = \vec{CB}$

$\vec{CA} + \vec{AB} = -\vec{BC}$ — (4)

By (3) & (4),

$\vec{BE} + \vec{DC} = 2\vec{BC} + \frac{1}{2}(-\vec{BC})$

$= \frac{3}{2}\vec{BC}$



Example 2. *Find a point O inside a quadrilateral ABCD such that if a particle placed at it be acted upon by forces represented by OA, OB, OC, OD, it will be in equilibrium.*

Let O be any point inside quad. ABCD & Let's assume E and F are mid points of AB & CD respectively.

Let G is mid point of EF.

In $\triangle OAB$, by λ-μ Theorem

$$\mu\vec{OA} + \lambda\vec{OB} = (\lambda+\mu)\vec{OE}$$

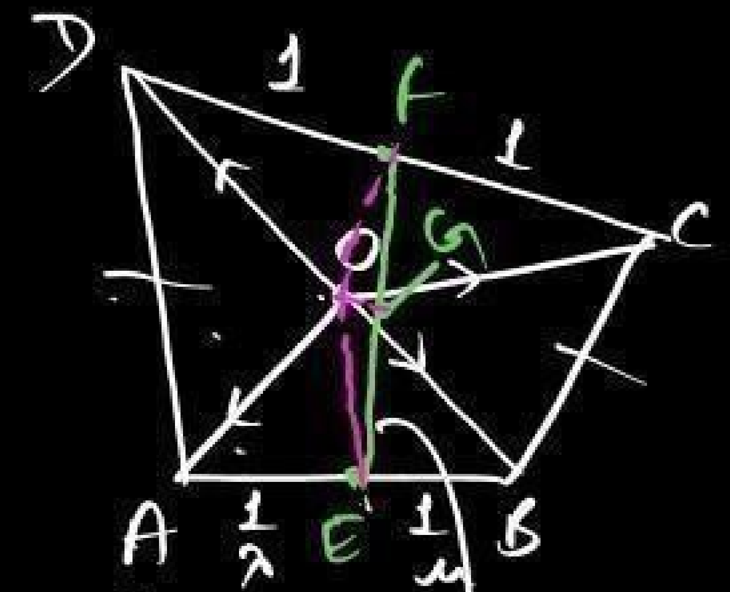
$$\Rightarrow 1.\vec{OA} + 1.\vec{OB} = 2\vec{OE} \quad \text{--- (1)}$$

In $\triangle OCD$, by λ-μ Theorem

$$1.\vec{OC} + 1.\vec{OD} = 2\vec{OF} \quad \text{--- (2)}$$

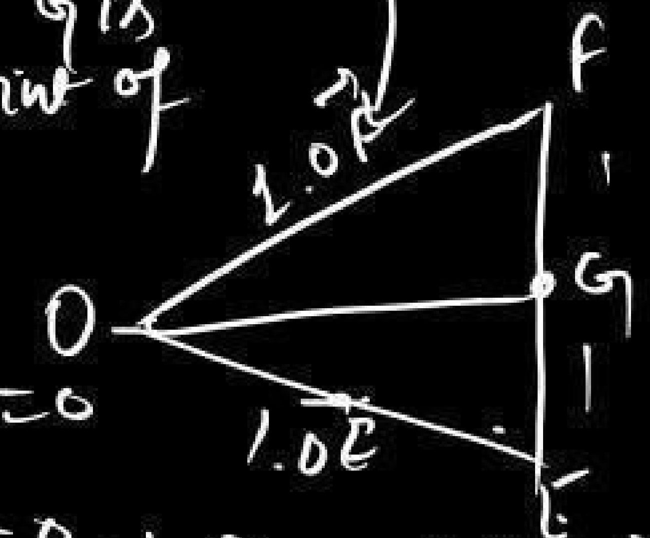
(1) + (2)

$$\vec{OA}+\vec{OB}+\vec{OC}+\vec{OD} = 2(\vec{OE}+\vec{OF}) = 4\vec{OG}$$, where G is mid point of EF



Since the point O is in Equilibrium.

$$\Rightarrow \vec{OA}+\vec{OB}+\vec{OC}+\vec{OD} = 0$$



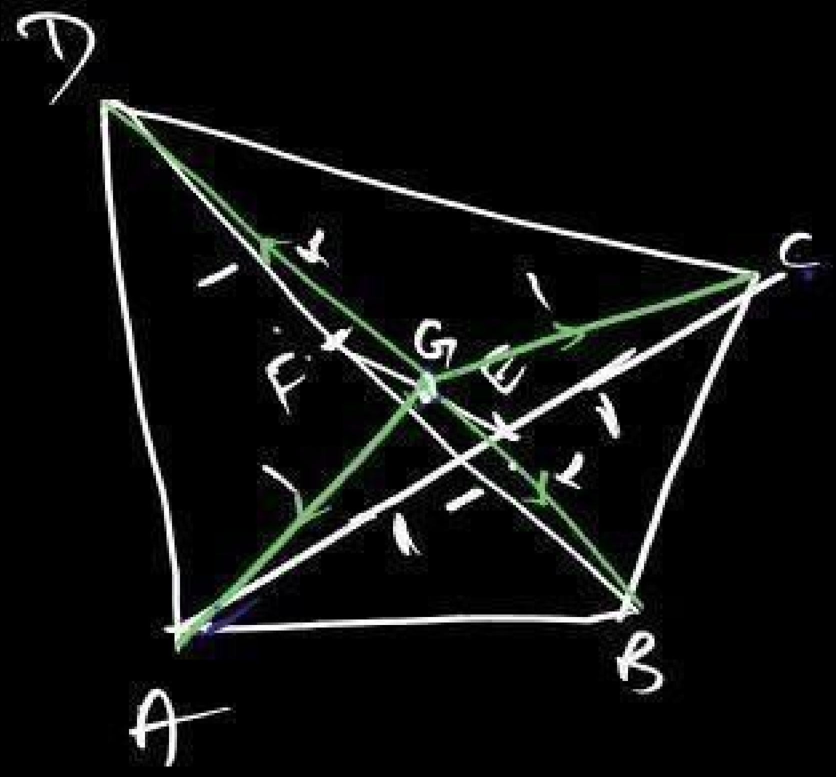
$$\Rightarrow 4\vec{OG} = 0 \Rightarrow \vec{OG} = 0 \Rightarrow$$ O and G coincides

$\therefore$ O is mid point of the line joining mid points of opposite sides of the quadrilateral

Example 5. *E, F are the mid-points of the diagonals AC and BD of a quadrilateral ABCD. If G is the mid-point of EF, show that the forces represented by GA, GB, GC, GD are in equilibrium.*

H.W.

In $\triangle AGC$; By λ-μ theorem.



$$1.GA + 1.GC = (1+1)GE$$

$$\Rightarrow \vec{GA} + \vec{GC} = 2\vec{GE}. \quad \text{—— (1)}$$

Similarly in $\triangle BDG$; by λ-μ theorem

$$1.\vec{GB} + 1.\vec{GD} = 2\vec{GF} \quad \text{—— (2)}$$

$$(1)+(2) \quad \vec{GA} + \vec{GB} + \vec{GC} + \vec{GD} = 2\vec{GE} + 2\vec{GF}$$

$$= 2(\vec{GE} + \vec{GF})$$

$$= 2(\vec{GE} - \vec{FG}) \quad (\because G \text{ is mid point of } EF)$$

$$= 2(\vec{GE} - \vec{GE})$$

$$= 0$$

Example 6. *A transversal cuts the lines of action of three concurrent forces P, Q, R in L, M, N respectively. If R is the resultant of P and Q, show that*

$$\frac{P}{OL} + \frac{Q}{OM} = \frac{R}{ON}$$

where O is the point of concurrence of the forces. [K.U. 2001]

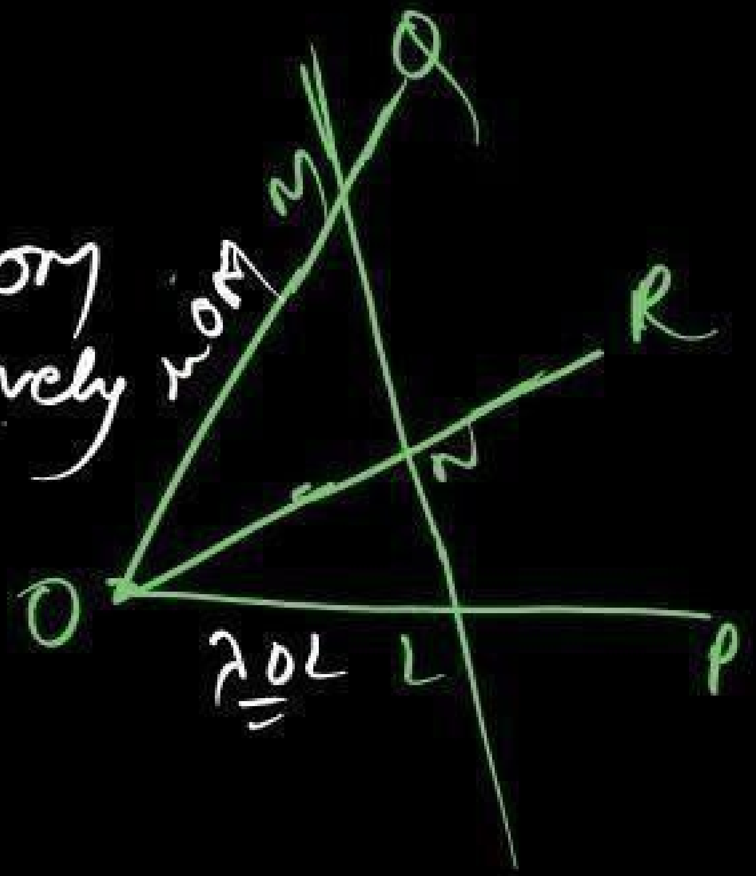
Let $\frac{P}{OL} = \lambda$ & let $\frac{Q}{OM} = \mu$

Now R is resultant of P & Q along OL & OM respectively w.r.t. O

By λ-μ theorem

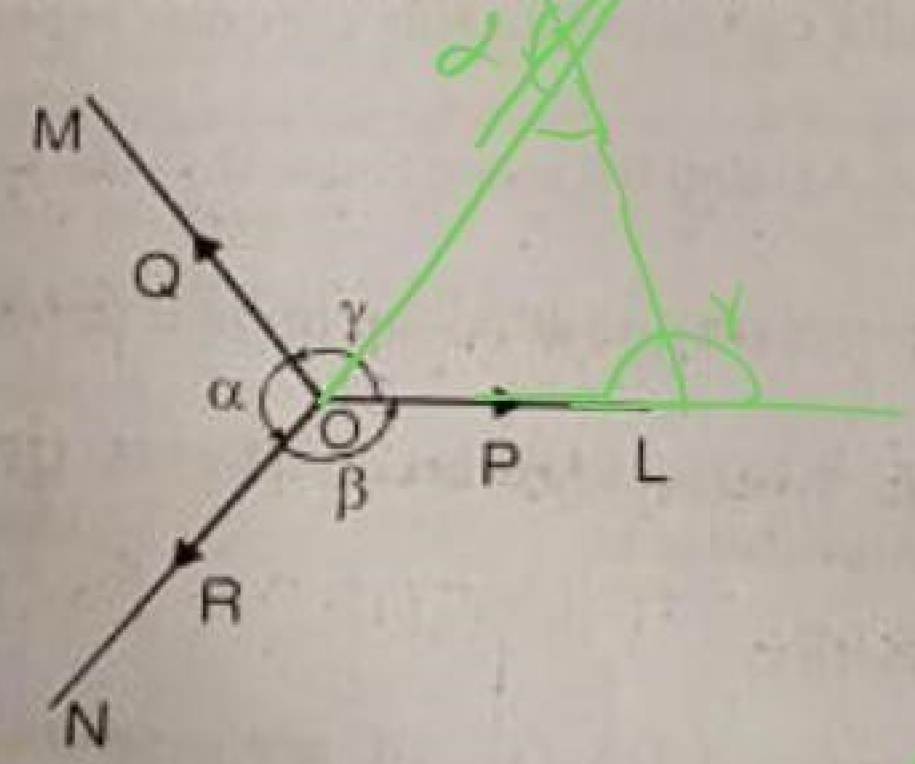
$R = (\lambda + \mu)\, ON$

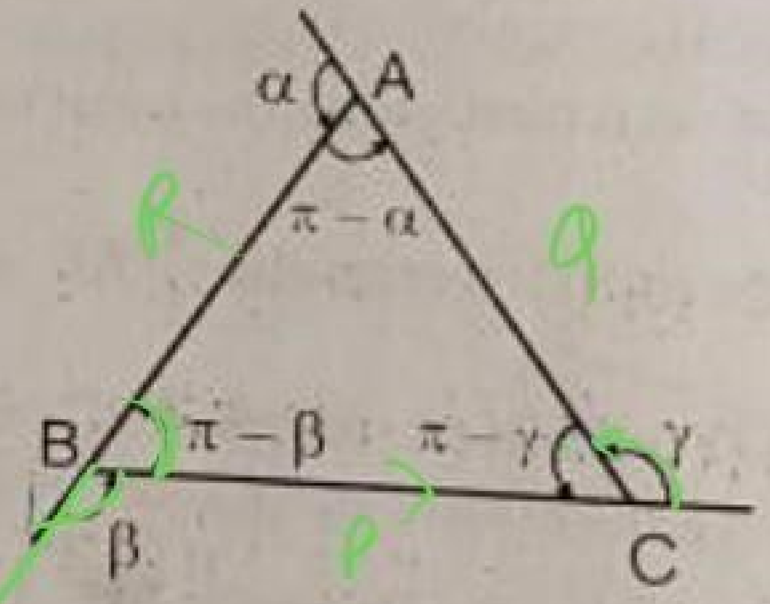
$\Rightarrow \frac{R}{ON} = \frac{P}{OL} + \frac{Q}{OM}$



1.11. LAMI'S THEOREM

Statement. *If three coplanar forces acting at a point are in equilibrium, then each is proportional to the sine of the angle between the other two.* [M.D.U. 2010]





$$\frac{P}{\sin(\pi-\alpha)} = \frac{Q}{\sin(\pi-\beta)} = \frac{R}{\sin(\pi-\gamma)}$$

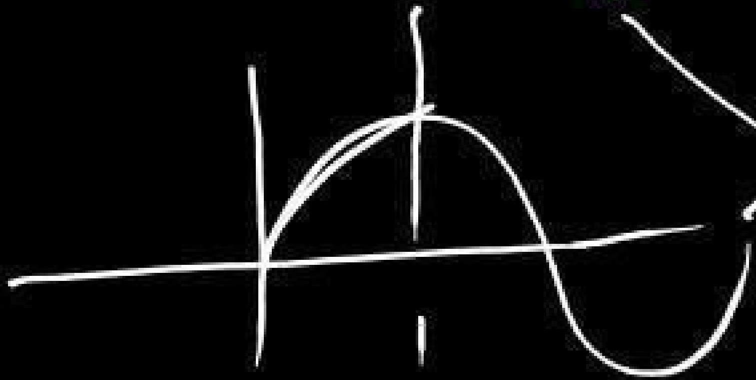
1.12. CONVERSE OF LAMI'S THEOREM

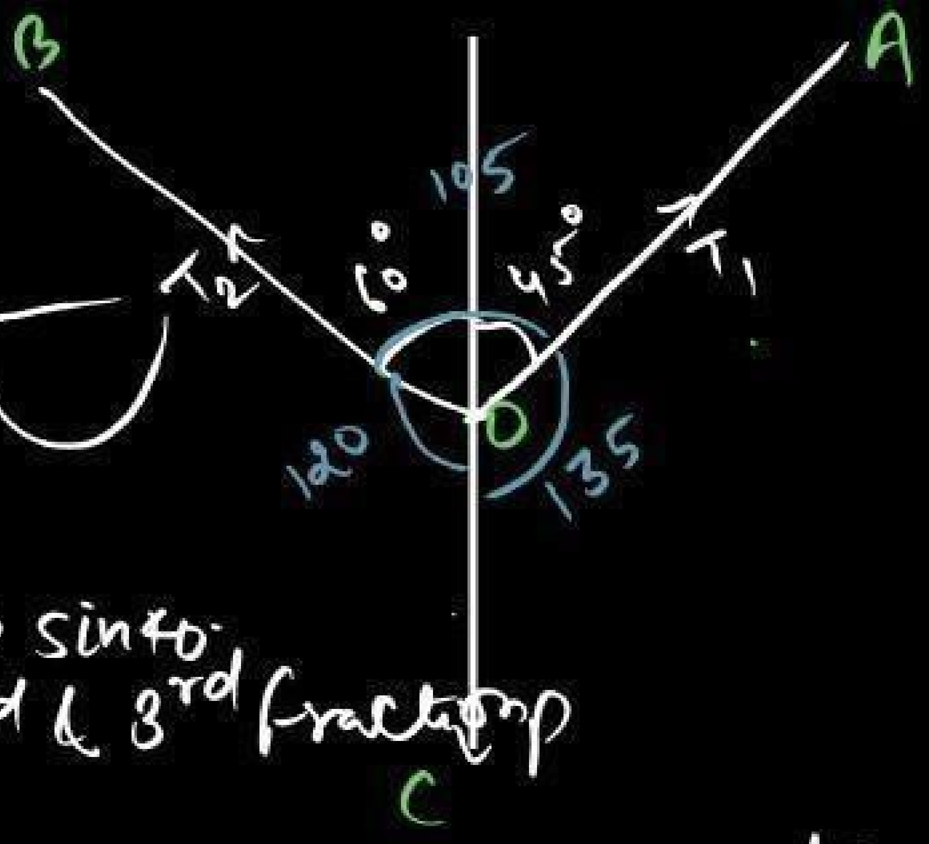
If three coplanar forces acting at a point be such that each is proportional to the sine of the angle between the directions of the other two, the forces are in equilibrium.

Example 2. *Find the greatest weight which can be supported by two light strings making angles 60° and 45° with the vertical; it being known that either string will break under the tension of W kg. wt.*

Let P be the greatest wt. which can be supported by two strings OA & OA.

According to Lami's theorem





$$\frac{P}{\sin\angle AOB} = \frac{T_1}{\sin\angle BOC} = \frac{T_2}{\sin\angle AOC}$$

$$\frac{P}{\sin(105)} = \frac{T_1}{\sin 120} = \frac{T_2}{\sin 135}$$

$$\frac{P}{\sin 75} = \frac{T_1}{\sin 60} = \frac{T_2}{\sin 45}$$

Since $\sin 60 > \sin 45$
then from 2nd & 3rd fraction
$T_1 > T_2$

$\therefore$ T_1 would be equal to W earlier.

Let $T_1 = W$

$$\frac{P}{\sin(30+45)} = \frac{W}{\sqrt{3}/2}$$

$$P = \frac{2}{\sqrt{3}}(\sin 30 \cos 45 + \cos 30 \sin 45) \times W = \frac{(1+\sqrt{3})}{\sqrt{3} \times 2\sqrt{2}} = (1+\sqrt{3})/\sqrt{6}\, W$$

Example 3. *Three forces P, Q, R acting at a point O are in equilibrium and the angle between P and Q is double the angle between P and R. Show that $R^2 = Q(Q - P)$.* [K.U. 2010]

$$\frac{P}{\sin(360-3\alpha)} = \frac{Q}{\sin\alpha} = \frac{R}{\sin 2\alpha}$$

$$\frac{P}{-\sin 3\alpha} = \frac{Q}{\sin\alpha} = \frac{R}{\sin 2\alpha}$$

$$P = -\frac{(3\sin\alpha - 4\sin^3\alpha)}{\sin\alpha}Q$$

$$P = -3Q + 4Q\sin^2\alpha$$

$$P + 3Q = 4Q(1-\cos^2\alpha)$$

$$P + 3Q = 4Q\left(1 - \frac{R^2}{4Q^2}\right)$$

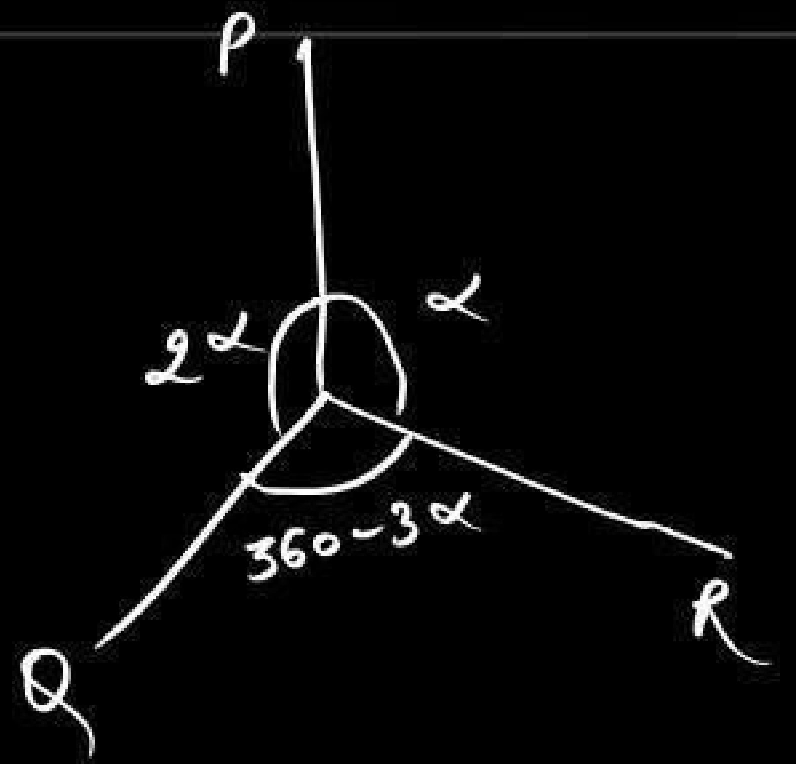
$$Q = \frac{\cancel{\sin\alpha}\, R}{2\cancel{\sin\alpha}\cos\alpha}$$

$$R = 2Q\cos\alpha$$

$$P + 3Q = \cancel{4}\cancel{Q}\frac{(4Q^2 - R^2)}{\cancel{4Q^2}}$$

$$Q(P+3Q) = 4Q^2 - R^2 \Rightarrow R^2 = 4Q^2 - QP - 3Q^2$$

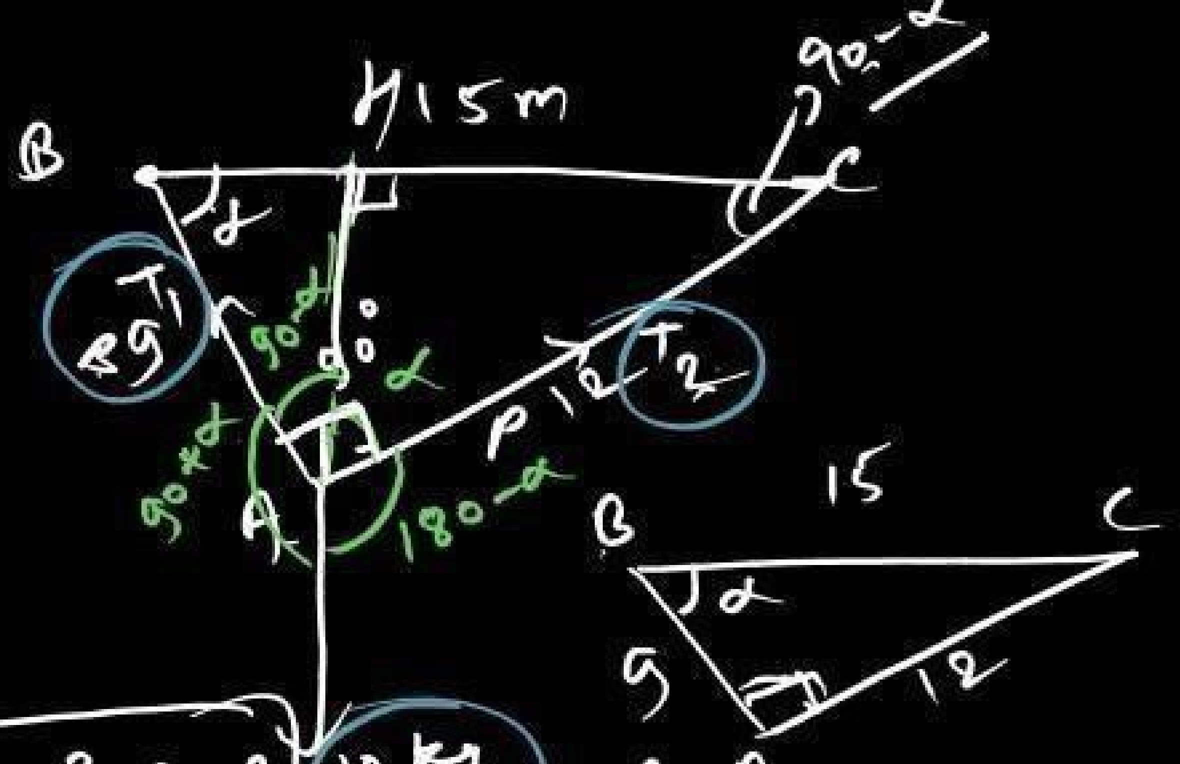
$$\Rightarrow R^2 = Q^2 - QP \Rightarrow R^2 = Q(Q-P)$$



Example 4. *AB and AC are two strings 9 m. and 12 m. long attached to pegs B and C at a horizontal distance 15 m apart. Find the tensions in the strings when a weight of 10 kg is suspended from A.*

By Lami's theorem over A

$$\cos\alpha = \frac{9}{15} = \frac{3}{5}$$



$$\frac{10}{\sin 90^\circ} = \frac{T_1}{\sin(180-\alpha)} = \frac{T_2}{\sin(90+\alpha)}$$

$$\frac{10}{1} = \frac{T_1}{\sin\alpha} = \frac{T_2}{\cos\alpha}$$

$$\cos\alpha = \frac{(15)^2+(9)^2-(12)^2}{2\times 15\times 9}$$

$$\cos\alpha = \frac{225+81-144}{270}$$

$$= \cancel{\frac{162}{270}}\ \cancel{\frac{81}{135}}$$

$$\boxed{\cos A = \frac{b^2+c^2-a^2}{2bc}}$$

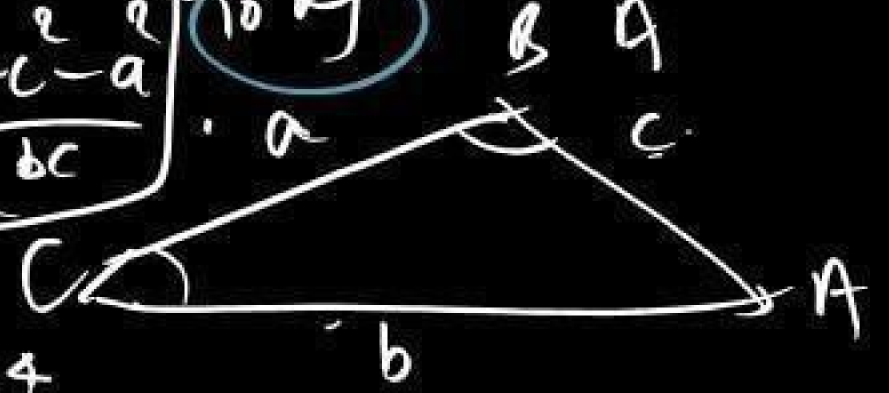
$$= \cancel{\frac{9}{15}}\ \frac{3}{5}$$

$\therefore T_1 = 10\sin\alpha$

$= 10\times\frac{4}{5} = 8$

$T_2 = 10\cos\alpha$

$= 10\times\frac{3}{5} = 6$



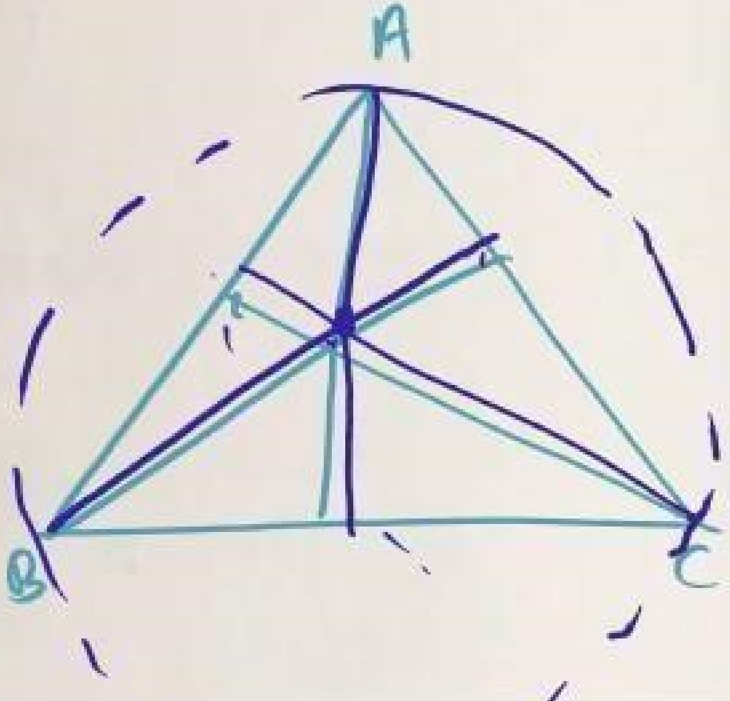
$\Rightarrow \cos\alpha = 3/5$, $\sin\alpha = \sqrt{1-\frac{9}{25}} = \frac{4}{5}$

Example 7. *ABC is a triangle and forces P, Q, R acting at a point O along the lines OA, OB, OC are in equilibrium. Prove that*

(a) *If O is the incentre of* ΔABC, *then* $\dfrac{P}{\cos\frac{A}{2}} = \dfrac{Q}{\cos\frac{B}{2}} = \dfrac{R}{\cos\frac{C}{2}}$

(b) *If O is the orthocentre of the* ΔABC, *then* $P : Q : R = a : b : c$

(c) *If O is the centroid of* ΔABC, *then* $\dfrac{P}{OA} = \dfrac{Q}{OB} = \dfrac{R}{OC}$.



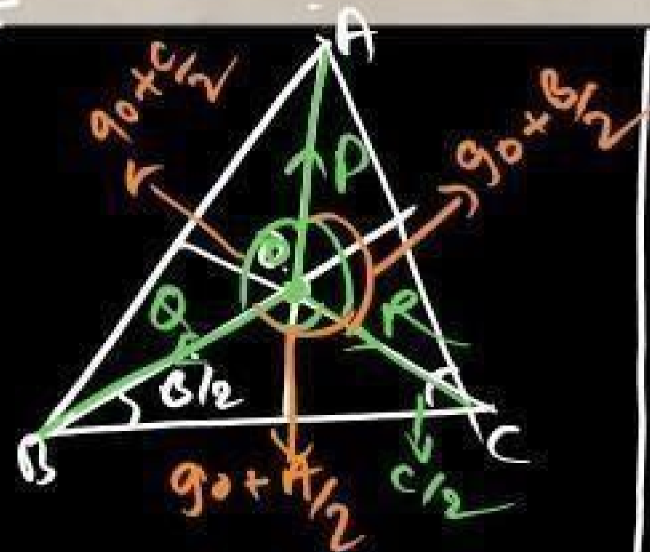
In $\triangle OBC$.

$\angle BOC + \frac{B}{2} + \frac{C}{2} = 180^\circ$

$\angle BOC = 180 - \left(\frac{B+C}{2}\right)$

$= 180 - \frac{(180-A)}{2}$

$= 90 + A/2$



By Lami's theorem $\dfrac{P}{\sin(90+A/2)} = \dfrac{Q}{\sin(90+B/2)} = \dfrac{R}{\sin(90+C/2)}$

$\Rightarrow \dfrac{P}{\cos A/2} = Q/\cos B/2 = R/\cos C/2$

meeting point of medians = Centroid.

meeting point of internal angle bisector = incentre

meeting points of altitudes = orthocentre

In $\triangle OBCE$

$\angle BEC = 90^\circ$

$\therefore \angle C + \angle EBC = 90^\circ$

$\angle EBC = 90^\circ - C$

then In $\triangle OBD$

$\angle 90 - C + 90^\circ + \angle BOD = 180^\circ$

$\cancel{90} - C + \cancel{90}\ \angle BOD = \cancel{180}$

$\angle BOD = C$

By Lami's Theorem

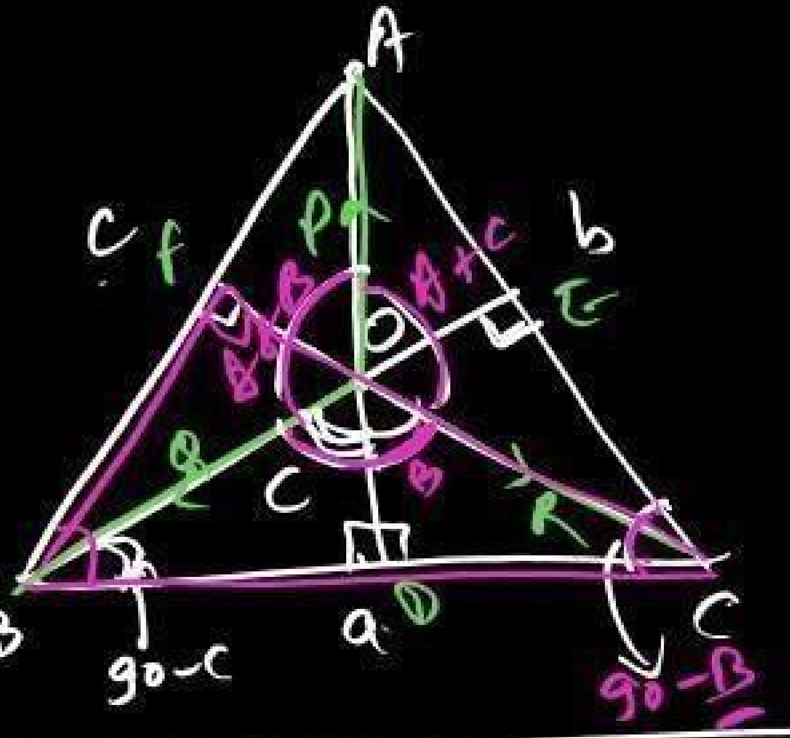
$$\frac{P}{\sin(B+C)} = \frac{Q}{\sin(A+C)} = \frac{R}{\sin(A+B)}$$

$$\frac{P}{\sin(180-A)} = \frac{Q}{\sin(180-B)} = \frac{R}{\sin(180-C)}$$

$$\frac{P}{\sin A} = \frac{Q}{\sin B} = \frac{R}{\sin C}$$

By sine formula for sides of a $\triangle$.

$$\frac{P}{a} = \frac{Q}{b} = \frac{R}{c}$$



Formulas

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

$$\cos B = \frac{a^2 + c^2 - b^2}{2ac}$$

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

Lami's theorem:-

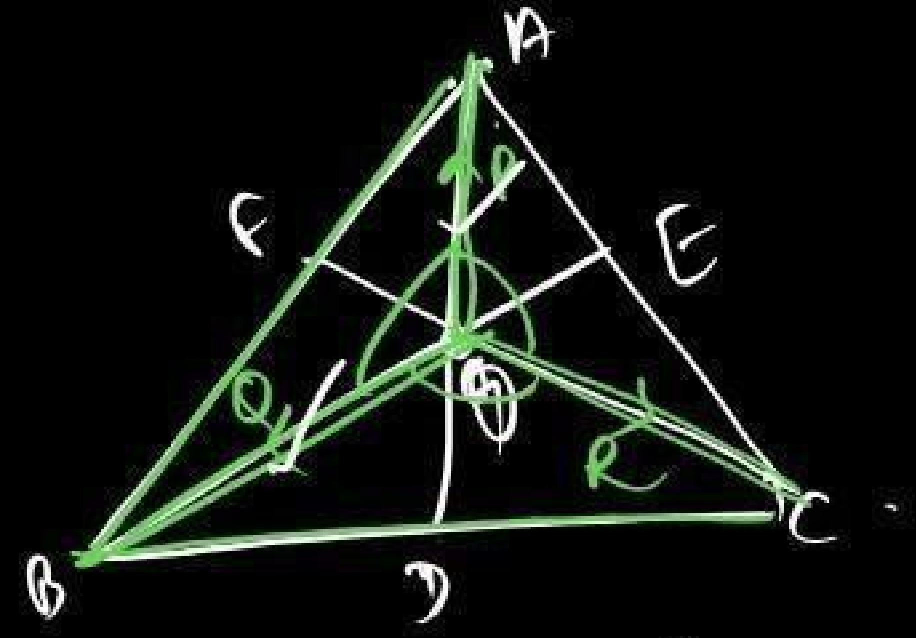
$$\frac{P}{\sin\angle BOC} = \frac{Q}{\sin\angle AOC} = \frac{R}{\sin\angle AOB} \quad \text{—— (1)}$$

Dividing (2) throughout by OA.OB.OC

$$\frac{\sin\angle AOB}{OC} = \frac{\sin\angle BOC}{OA} = \frac{\sin\angle AOC}{OB} \quad \text{—— (2)}$$

By (1) & (2)

$$\frac{P}{OA} = \frac{Q}{OB} = \frac{R}{OC}$$



$$2OD = OA$$

$$\frac{AO}{OD} = \frac{2}{1}$$

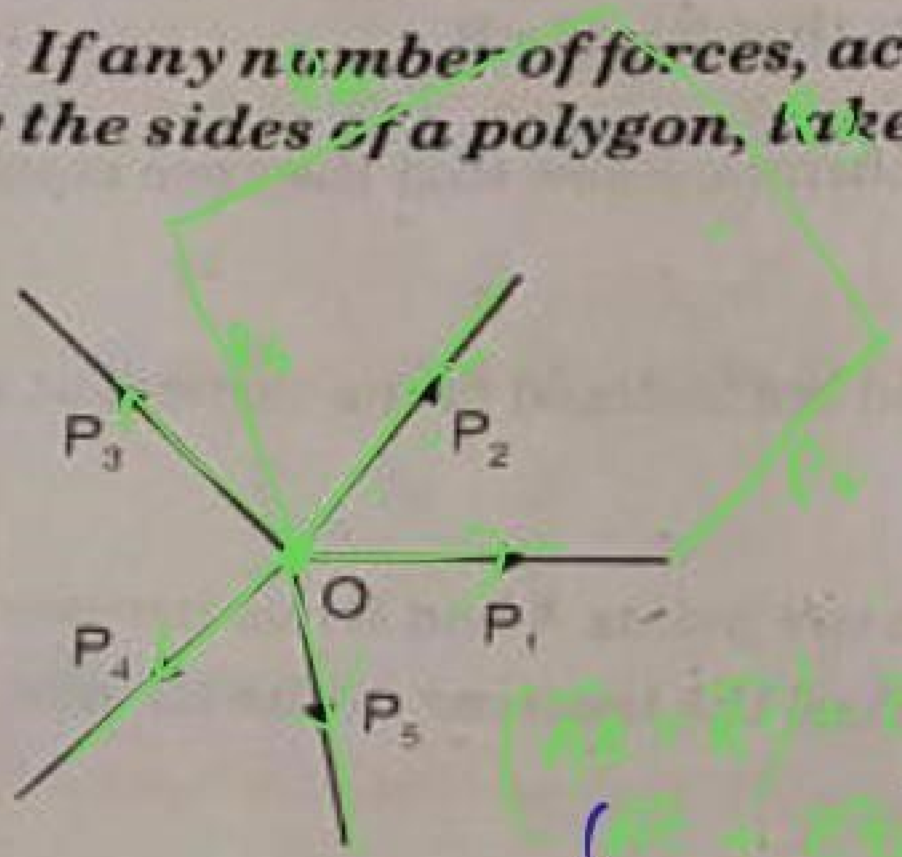
$$\Delta AOB = \Delta BOC = \Delta AOC$$

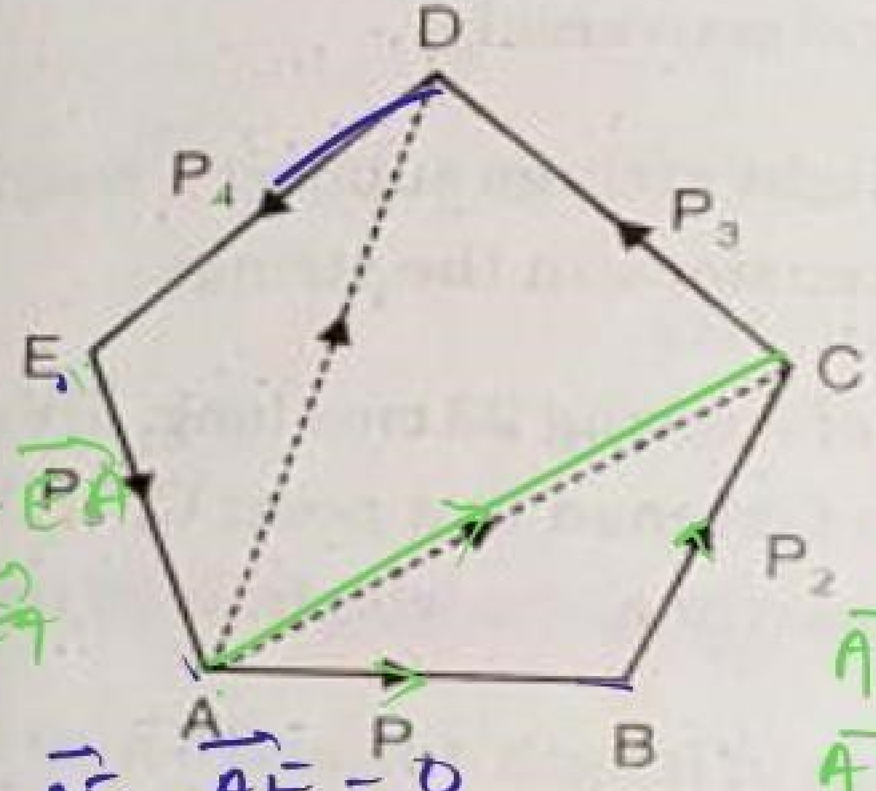
$$\cancel{\tfrac{1}{2}}\, OA.OB \sin\angle AOB = \cancel{\tfrac{1}{2}}\, OB.OC \sin\angle BOC$$

$$= \cancel{\tfrac{1}{2}}\, OA\, OC \cdot \sin\angle AOC \quad \text{—— (2)}$$

1.13. POLYGON LAW OF FORCES

Statement. *If any number of forces, acting at a point, be represented in magnitude and direction by the sides of a polygon, taken in order, the forces will be in equilibrium.*





$(\vec{AB}+\vec{BC})+\vec{CD}+\vec{DE}+\vec{EA}$

$(\vec{AC}+\vec{CD})+\vec{DE}+\vec{EA}$

$(\vec{AD}+\vec{DE})+\vec{EA}$

$\vec{AE}+\vec{EA} = \vec{AE}-\vec{AE} = 0$

$\vec{AB}+\vec{BC} = \vec{AC}$ — (1)

In $\triangle ACD$

$\vec{AC}+\vec{CD} = \vec{AD}$ — (2)

$\vec{AD}+\vec{DE} = \vec{AE}$ — (3)

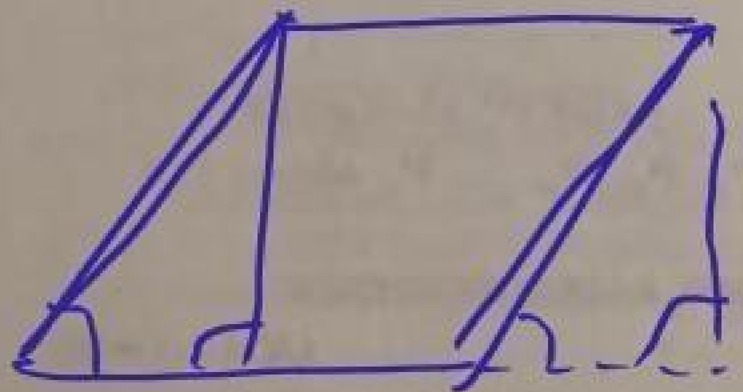
Hence the Polygon Law of forces can also be stated as :

If any number of forces, acting at a point, can be represented in magnitude and direction by the sides, taken in order, of an open polygon, their resultant is represented in magnitude and direction by the closing side of the polygon in the opposite order.

1.14. THEOREM OF RESOLVED PARTS

Statement. ***The algebraic sum of the resolved parts of two concurrent forces in any direction in their plane is equal to the resolved part of their resultant in the same direction.***

Proof. Let P and Q be the two given forces represented by OA and OB respectively. Complete the parallelogram OACB. Their resultant R is represented by the diagonal OC.



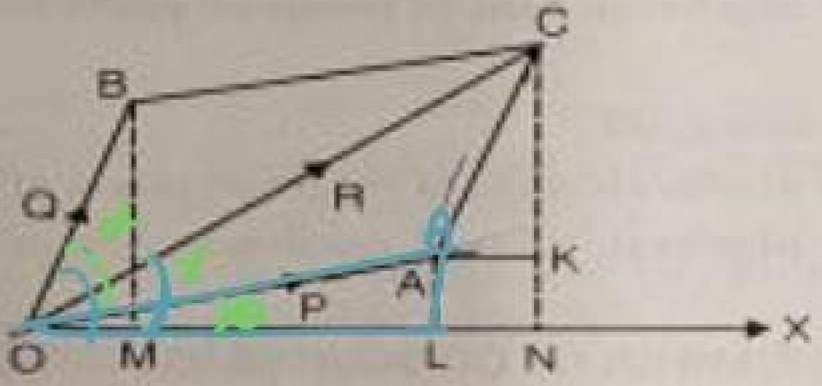


Fig. 1.34

P cos θ

Q cos(α+θ)

Let OX be parallel to the direction in which the forces are to be resolved.

Draw AL, BM and CN ⊥'s to OX and AK ⊥ CN. Clearly △'s OMB and AKC are congruent.

$$\therefore \quad OM = AK$$

$$\Rightarrow \quad OM = LN \qquad \ldots(1) \quad [\because AK = LN]$$

⇒ OM

Now, resolved part of P along OX

$$= P \cos \angle AOL$$

$$= OA \cdot \frac{OL}{OA} = OL$$

ON = OL + LN

Now △OBM ≅ △ACK

OM = AK

= LN

Similarly, the resolved part of Q along OX = OM

and the resolved part of R along OX = ON

∴ The algebraic sum of the resolved parts of P and Q along OX

$$= OL + OM = OL + LN$$

$$= ON = \text{Resolved part of R along OX} \qquad [\text{From } (1)]$$

ON = OL + OM

Res part of R =

Hence the theorem.

⇒ Sum of resolved part of P & Q along x

(P cos θ + Q cos(α+θ)) — ①

Resolved part of R = R cos γ — ④

In △OAL

cos θ = OL/P — ②

In △OBM

cos(α+θ) = OM/OB = OM/Q — ③

In △OCN:

cos γ = ON/OC = ON/R — ④

By ①, ② & ③

Sum of R.P. of P & Q

= P × OL/P + Q × OM/Q

= OL + OM

From ④ & ④ Res. P. of R = ON

Hence the theorem.

1.16. RESULTANT OF ANY NUMBER OF CONCURRENT AND COPLANAR FORCES

[K.U. 1995]

Let $P_1, P_2, P_3, \ldots, P_n$ be n coplanar forces acting at a point O in directions making angles $\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n$ respectively with a fixed straight line OX lying in the plane of the forces.

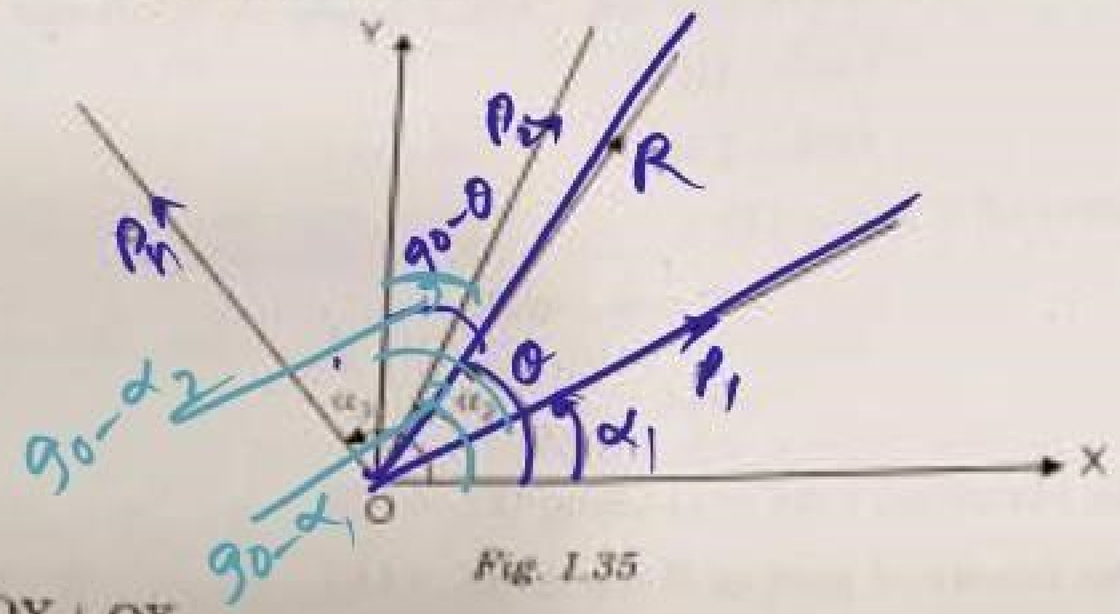


Fig. 1.35

Through O draw OY ⊥ OX.

Let their resultant R make an angle θ with OX. Now by the Generalised Theorem of resolved parts :

Resolving along OX, we have

$$R \cos \theta = P_1 \cos \alpha_1 + P_2 \cos \alpha_2 + P_3 \cos \alpha_3 + \ldots + P_n \cos \alpha_n \qquad \ldots(1)$$
$$= X \text{ (say)}$$

Resolving along OY, we have

$$R \sin \theta = P_1 \sin \alpha_1 + P_2 \sin \alpha_2 + P_3 \sin \alpha_3 + \ldots + P_n \sin \alpha_n \qquad \ldots(2)$$
$$= Y \text{ (say)}$$

Squaring (1) and (2) and adding, we have

$$R^2 = X^2 + Y^2$$

$$R = \sqrt{X^2 + Y^2}, \text{ which is the magnitude of the resultant.}$$

Along y-axis.

$R \cos(90-\theta) = P_1 \cos(90-\alpha_1) + P_2 \cos(90-\alpha_2) + \ldots + P_n \cos(90-\alpha_n)$

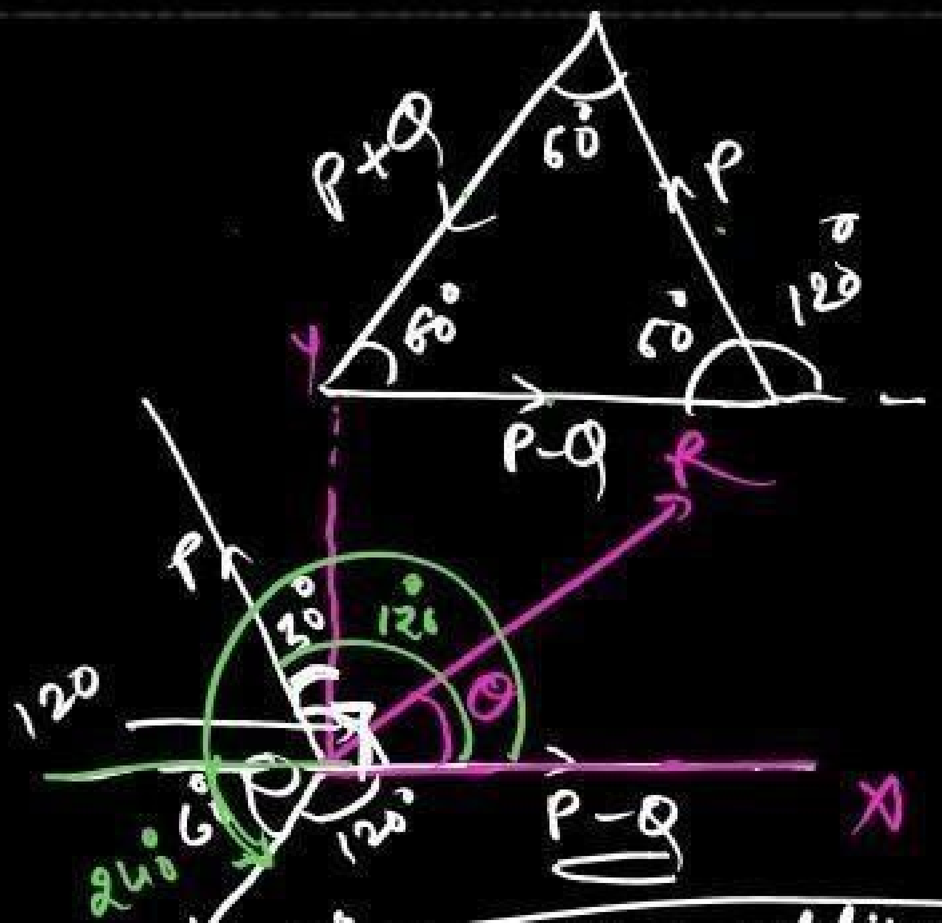
$R \sin\theta = P_1 \sin\alpha_1 + P_2 \sin\alpha_2 + \ldots + P_n \sin\alpha_n$.

Example 1. *Forces $P - Q$, P, $P + Q$ act at a point in direction parallel to the sides of an equilateral triangle, taken in order. Find their resultant.*

Let R is the resultant of $P-Q$, P, $P+Q$. &

it makes an angle θ with $P-Q$

Now A.T. Resolved part theorem



Along $\underline{OX}$

$$R\cos\theta = (P-Q)\cos 0° + P\cos 120° + (P+Q)\cos 240°$$

$$= P-Q + P\cos(180-60) + (P+Q)\cos(180+60°)$$

$$= P-Q - P\cos 60° - (P+Q)\cos 60°$$

$$= P-Q - \frac{P}{2} - \frac{(P+Q)}{2} = -\frac{3Q}{2} \quad \text{—— (1)}$$

Along OY :— $R\sin\theta = (P-Q)\sin 0 + P\sin(180-60) + (P+Q)\sin(180+60)$

$$R\sin\theta = \frac{\sqrt{3}}{2}P - (P+Q)\frac{\sqrt{3}}{2} = -Q\frac{\sqrt{3}}{2} \quad \text{— (2)}$$

Squaring & adding (1) & (2)

$$R^2(\cos^2\theta + \sin^2\theta) = \frac{9}{4}Q^2 + \frac{3}{4}Q^2$$

$$R^2 = 3Q^2$$

$$\boxed{R = \sqrt{3}\,Q}$$

Example 3. *ABCDEF is a regular hexagon. Forces of magnitude* $4, 8\sqrt{3}, 16, 4\sqrt{3}$ *and 8 Newtons act at A in the directions AB, AC, AD, AE and AF respectively. Find the resultant of the forces.*

In $\triangle ABC$

$AB = BC$

$\therefore \angle CAB = \angle ACB$

$\therefore \underbrace{\angle CAB + \angle ACB} + \angle ABC = 180$

$2\angle CAB + 120 = 180$

$\Rightarrow \boxed{\angle CAB = 30}$

Resolving the forces Along AX

$R\cos\theta = 4\cos 0^\circ + 8\sqrt{3}\cos 30^\circ + 16\cos 60^\circ + 4\sqrt{3}\cos 90^\circ + 8\cos 120$

$= 4 + 8\sqrt{3} \times \frac{\sqrt{3}}{2} + \frac{16}{2} - 8 \times \frac{1}{2}$

$= 12 + 8 = 20$ — (1)

Resolving along AY

$R\sin\theta = 4\sin 0^\circ + 8\sqrt{3}\sin 30^\circ + 16\sin 60^\circ + 4\sqrt{3}\sin 90^\circ + 8\sin(180-60)$

$= 8\sqrt{3} \times \frac{1}{2} + 16\frac{\sqrt{3}}{2} + 4\sqrt{3} + 8\sqrt{3}/2$

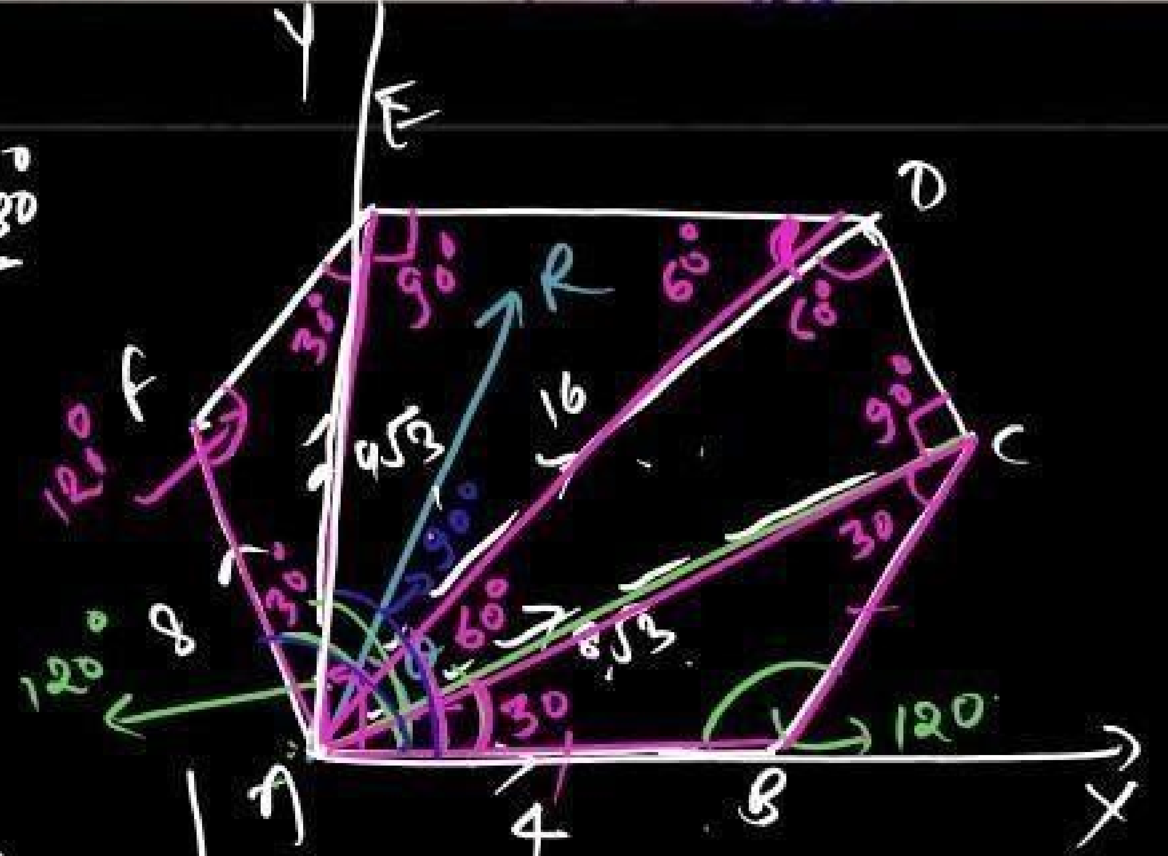
$= 20\sqrt{3}$ — (2)

Squaring & adding (1) & (2)

$R^2 = (20)^2 + (20\sqrt{3})^2$

$= 400 + 1200 = 1600$

$R = 40$



In a polygon of n sides.
Angle sustended by two adjacent sides $= \left(\frac{n-2}{n}\right)\pi$

$\frac{6-2}{6} \times 180$

$= \frac{4}{6} \times 180$

$= 120$

Example 4. *ABCDEF is a regular hexagon. The forces represented in magnitude and direction by AB, 2AC, 3AD, 4AE, 5AF act at A. Show that the magnitude of the resultant is $\sqrt{351}$ AB and its direction is inclined at an angle $\tan^{-1}\dfrac{7}{\sqrt{3}}$ to AB.*

$\dfrac{AL}{AB} = \cos 30^\circ$

$AL = \dfrac{a\sqrt{3}}{2}$; $AC = 2AL = \sqrt{3}a$

$AD = \sqrt{3a^2 + a^2} = 2a$

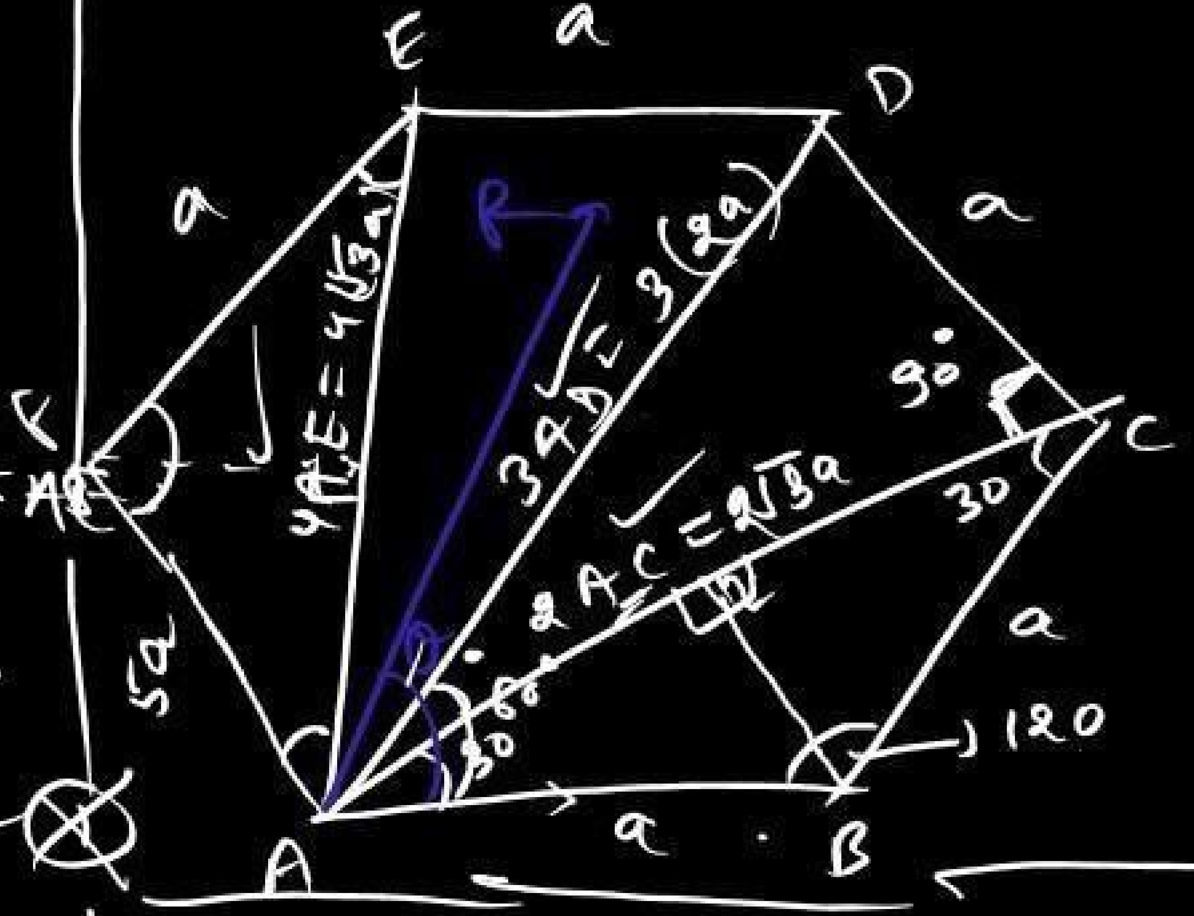
$AE = \sqrt{3}a$

Resolving all the forces along AB & AE

Along AB :—

$R\cos\theta = a\cos 0^\circ + 2\sqrt{3}a\cos 30 + 6a\cos 60^\circ + 4\sqrt{3}a\cos 90 + 5a\cos 120$

$R\cos\theta = a + 3a + 3a - \dfrac{5a}{2} = \dfrac{9a}{2}$ — (1)



$AC = 2AL$

$\dfrac{AL}{AB} = \cos 30^\circ$

$AL = a\cos 30^\circ$

$= \dfrac{\sqrt{3}}{2}a$

Resolving all the forces along AE

$R\sin\theta = a\sin 0^\circ + 2\sqrt{3}a\sin 30^\circ +$
$6a\sin 60^\circ + 4\sqrt{3}a\sin 90^\circ + 5a\sin 120^\circ$

$R\sin\theta = \sqrt{3}a + \cancel{6}^{3}a\frac{\sqrt{3}}{\cancel{2}} + 4\sqrt{3}a + 5a\frac{\sqrt{3}}{2}.$

$= 8\sqrt{3}a + 5a\frac{\sqrt{3}}{2}$

$R\sin\theta = \frac{21\sqrt{3}}{2}a$ —— (2)

$(1)^2 + (2)^2$

$R^2(\cos^2\theta + \sin^2\theta) = \frac{81a^2}{4} + \frac{\overset{1323}{441\times 3}a^2}{4}$

$R^2 = \frac{1404}{4}a^2 = 351a^2$

$R = \sqrt{351}\,a$
$= \sqrt{351}\,AB$

$\frac{(2)}{(1)} \Rightarrow \frac{R\sin\theta}{R\cos\theta} = \frac{\overset{7}{\cancel{21}}\sqrt{3}\cancel{a}\times\cancel{2}}{\cancel{2}\times\underset{3}{\cancel{9}}\cancel{a}}$

$\Rightarrow \tan\theta = \frac{7}{\sqrt{3}}$

$\theta = \tan^{-1}\left(\frac{7}{\sqrt{3}}\right)$

1.17. Working rule for solving problems on equilibrium of a number of concurrent forces :

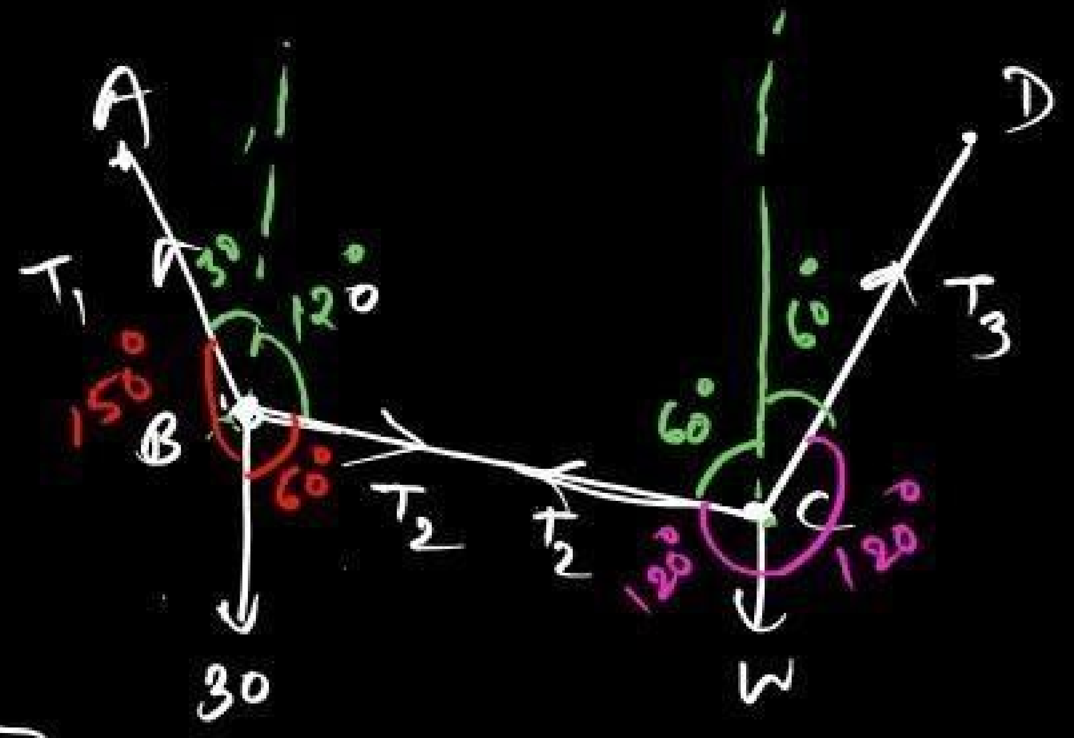
1. Choose two convenient directions at right angles to each other in the figure as axes, usually horizontal and vertical.
2. Resolve the forces along each of these directions and equate them separately equal to zero.
3. Solve these two equations and obtain the required result.

Note. For problems where there are only three forces acting on a particle, use **Lami's Theorem** to get the desired result

Example 1. *A string ABCD is suspended from two fixed points A and D. It carries weights of 30 kg and W kg respectively at two points B and C in it. The inclination to the vertical of AB is 30° and that of CD is 60°, the angle BCD being 120°. Find W and the tension in the different parts of the string.*

At B; 3 forces T_2, T_1 & 30kg are Acting

Applying Lami's theorem at B



$$\frac{T_1}{\sin 60^\circ} = \frac{30}{\sin 150^\circ} = \frac{T_2}{\sin 150^\circ}$$

$$\Rightarrow \frac{T_1}{\sin 60^\circ} = \frac{30}{\sin 30} = \frac{T_2}{\sin 30} \Rightarrow \frac{T_1}{\sqrt{3}/\cancel{2}} = \frac{30}{1/\cancel{2}} = \frac{T_2}{1/\cancel{2}}.$$

$$\Rightarrow \boxed{T_1 = 30\sqrt{3}, \; T_2 = 30}$$

Again applying Lami's theorem at C:

$$\frac{W}{\sin 120} = \frac{T_3}{\cancel{\sin 120}} = \frac{T_2}{\cancel{\sin 120}} \Rightarrow \boxed{W = T_3 = 30}$$

Example 2. *A string of length l is fastened to two points A, B at the same level and at a distance 'a' apart. A ring of weight W can slide on the string and a horizontal force P is applied to it such that it is in equilibrium vertically below B. Show that*

$$P = \frac{aW}{l} \text{ and tension of the string is } \frac{W(l^2+a^2)}{2l^2}.$$ [K.U. 1996; M.D.U. 1994]

Given $AC + BC = l$ & let $\angle ACB = \theta$

Let $BC = x$.

then $AC = l - x$

Resolving the forces →

Along CD

$P\cos 0^\circ + T\cos 90^\circ + T\cos(90+\theta) + W\cos 270 = 0$

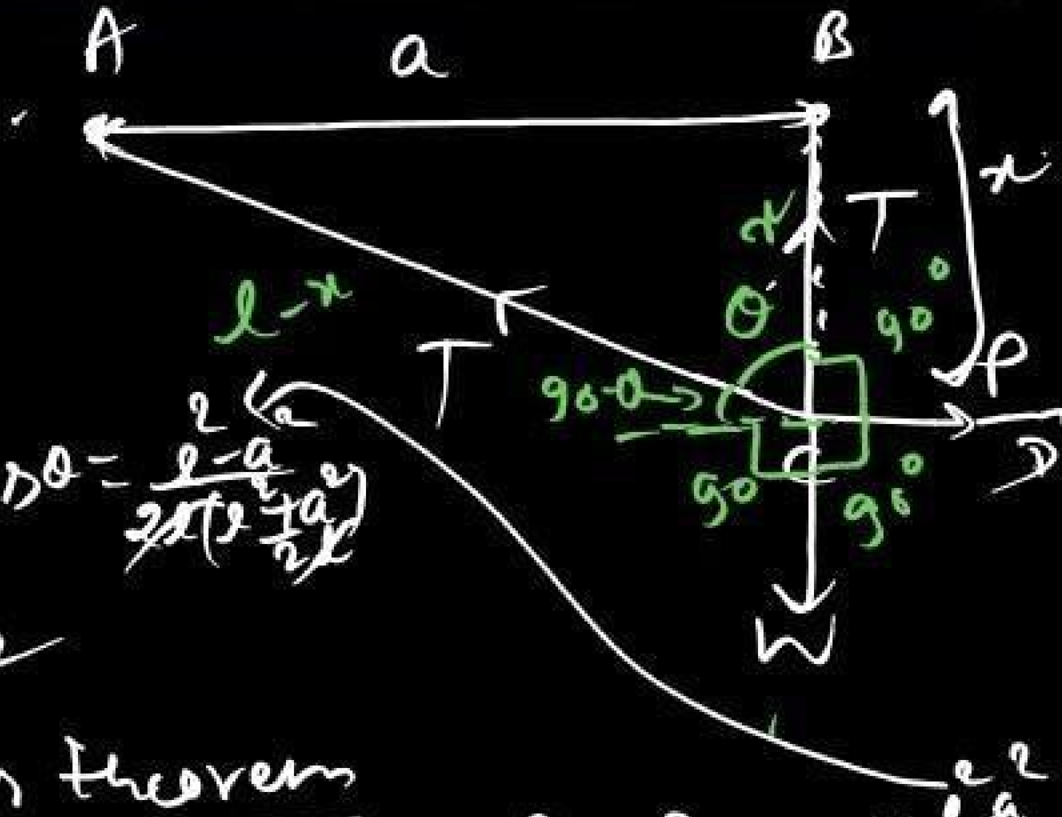
$P - T\sin\theta = 0 \Rightarrow \boxed{P = T\sin\theta}$

Along CB

$P\sin 0^\circ + T\sin 90^\circ + T\sin(90+\theta) + W\sin 270 = 0$

$T + T\cos\theta - W = 0$

$W = T(1+\cos\theta)$



In $\triangle ABC$

$\cos\theta = \frac{x}{l-x} \Rightarrow \cos\theta = \frac{\frac{l^2-a^2}{\cancel{2l}}}{\frac{l^2+a^2}{\cancel{2l}}}$

$\sin\theta = \frac{a}{l-x} = \frac{2la}{l^2+a^2}$

Also by phythagorous theorem

$(l-x)^2 = a^2 + x^2 \Rightarrow l^2 + \cancel{x^2} - 2lx = a^2 + \cancel{x^2} \Rightarrow x = \frac{l^2-a^2}{2l}$

$l - x = l - \frac{l^2-a^2}{2l} = \frac{l^2+a^2}{2l}$

$$W = T(1+\cos\theta)$$

$$T = \frac{W}{1+\frac{(l^2-a^2)}{l^2+a^2}}$$

$$\boxed{T = \frac{W(l^2+a^2)}{2l^2}}$$

$$P = T\sin\theta.$$

$$= \frac{W\cancel{(l^2+a^2)}}{\cancel{2}l^{\cancel{2}}} \times \frac{\cancel{2}al}{\cancel{l^2+a^2}}$$

$$\boxed{P = \frac{Wa}{l}}$$

Example 3. *Two weights P, Q (P > Q) attached to the ends of a string rest on a smooth circular disc whose plane is vertical. Prove that the inclination θ to the horizontal of the line joining them is given by,* $\boxed{\tan\theta = \dfrac{P-Q}{P+Q}\tan\alpha}$, *where 2α is the angle subtended by PQ at the centre.*

In $\triangle AOB$; $\angle A = \angle B$ (subtended by radius vector)

$\angle A + \angle O + \angle B = 180^\circ$

$2\angle A = 180 - 2\alpha$

$\angle A = 90 - \alpha$

And $\angle RAQ = 90^\circ$

$90 - \alpha + \theta + \angle OAQ = 90$

$\angle OAQ = \alpha - \theta$

At point A; Q, R & T are in eq^m

∴ By Lami's Theorem:—

$$\frac{T}{\sin(180+\theta-\alpha)} = \frac{R}{\sin(90+\alpha-\theta)} = \frac{Q}{\sin 90}$$

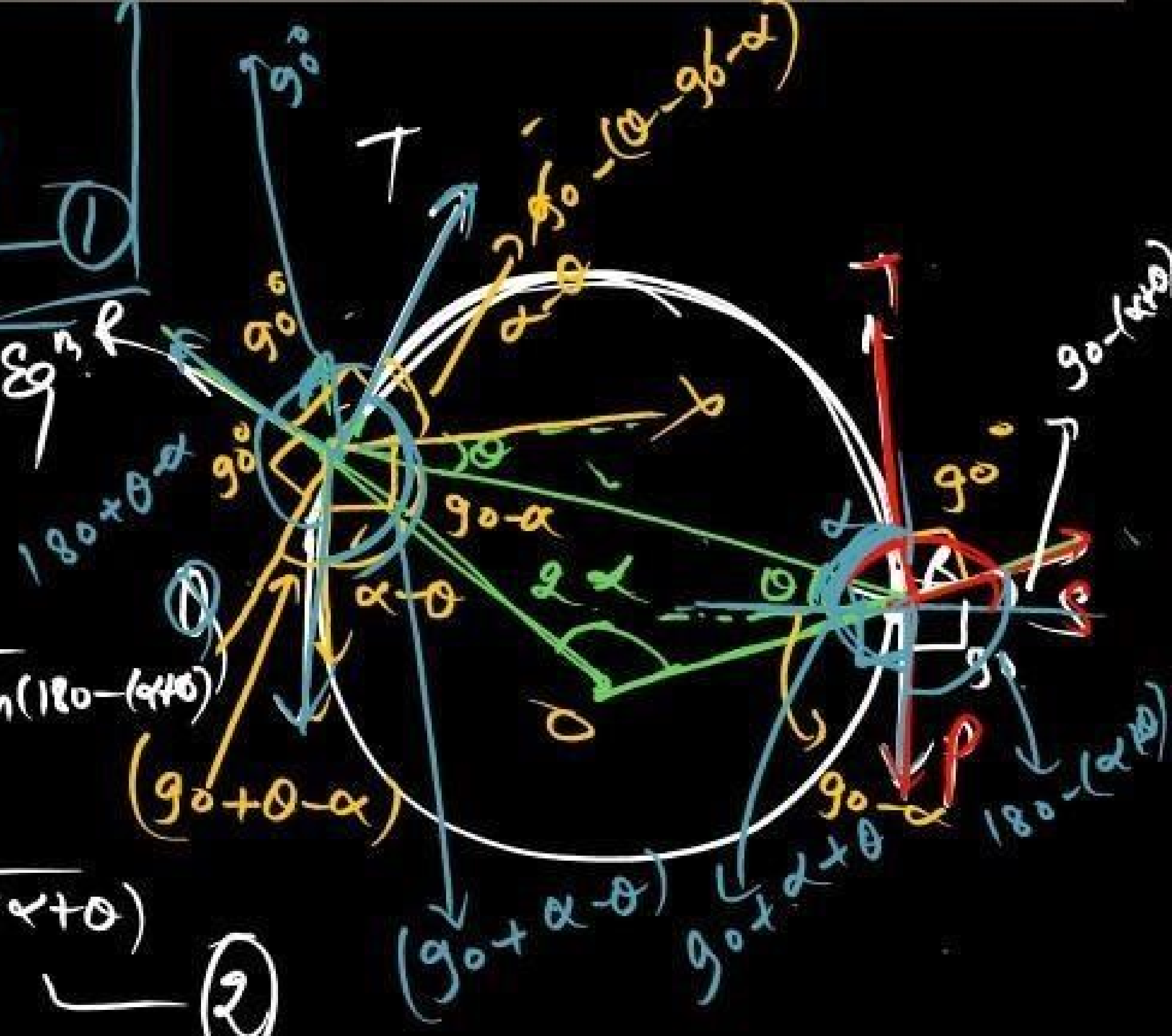
$$\frac{T}{\sin(\theta-\alpha)} = \frac{R}{\cos(\alpha-\theta)} = Q \quad (1)$$

At B; P, S & T are in Eq^n

∴ By Lami's Theorem

$$\frac{P}{\sin 90} = \frac{S}{\sin(90+\alpha+\theta)} = \frac{T}{\sin(180-(\alpha+\theta))}$$

$$\Rightarrow \frac{P}{1} = \frac{S}{\cos(\alpha+\theta)} = \frac{T}{\sin(\alpha+\theta)} \quad (2)$$



Componendo & Dividendo

If $\frac{A}{B} = \frac{C}{D}$.

then $\frac{A+B}{A-B} = \frac{C+D}{C-D}$

from Eq^n 1; considering Ist & 3rd fraction;

we get $T = -Q \sin(\theta - \alpha)$ — (3)

from Eq^n 2; considering Ist & 3rd fraction.

$$T = P \sin(\alpha + \theta) \text{ — (4)}$$

By (3) & (4)

$$Q \sin(-\theta + \alpha) = P \sin(\alpha + \theta) \quad (\because \sin(-\theta) = -\sin\theta)$$

$$\frac{P}{Q} = \frac{\sin(\alpha+\theta)}{\sin(\alpha-\theta)}$$

$$\frac{P-Q}{P+Q} = \frac{\sin(\alpha+\theta) - \sin(\alpha-\theta)}{\sin(\alpha+\theta) + \sin(\alpha-\theta)}$$

$$\frac{P-Q}{P+Q} = \frac{\cancel{\sin\alpha\cos\theta} + \cos\alpha\sin\theta - \cancel{\sin\alpha\cos\theta} + \cos\alpha\sin\theta}{\sin\alpha\cos\theta + \cancel{\cos\alpha\sin\theta} + \sin\alpha\cos\theta - \cancel{\cos\alpha\sin\theta}}$$

$$= \frac{\cancel{2}\cos\alpha\sin\theta}{\cancel{2}\sin\alpha\cos\theta}.$$

$$\boxed{\frac{P-Q}{P+Q}\tan\alpha = \tan\theta}$$

Example 5. *Two weights P and Q are suspended from a fixed point O by string OA, OB and are kept apart by a light rod AB. If the string makes angles α and β with the rod, show that the angle θ which the rod makes with the vertical is given by*

$$\tan\theta = \frac{P+Q}{P\cot\alpha - Q\cot\beta}.$$

At $\underline{\underline{B}}$; Q, R and T_2 are in Eq^m, ∴ By Lami's theorem

$$\frac{Q}{\sin(180-\beta)} = \frac{R}{\sin(\theta+\beta)} = \frac{T_2}{\sin(180-\theta)}$$

$$\frac{Q}{\sin\beta} = \frac{R}{\sin(\theta+\beta)} = \frac{T_2}{\sin\theta} \quad \text{— (1)}$$

At A; P, R & T_1 are in Eq^m.

∴ By Lami's theorem.

$$\frac{T_1}{\sin\theta} = \frac{R}{\sin(180-(\theta-\alpha))} = \frac{P}{\sin(180-\alpha)}$$

$$\Rightarrow \frac{T_1}{\sin\theta} = \frac{R}{\sin(\theta-\alpha)} = \frac{P}{\sin\alpha} \quad \text{— (2)}$$

from Eq^n (1) & (2)

$$R = Q\frac{\sin(\theta+\beta)}{\sin\beta} = \frac{P\sin(\theta-\alpha)}{\sin\alpha}$$

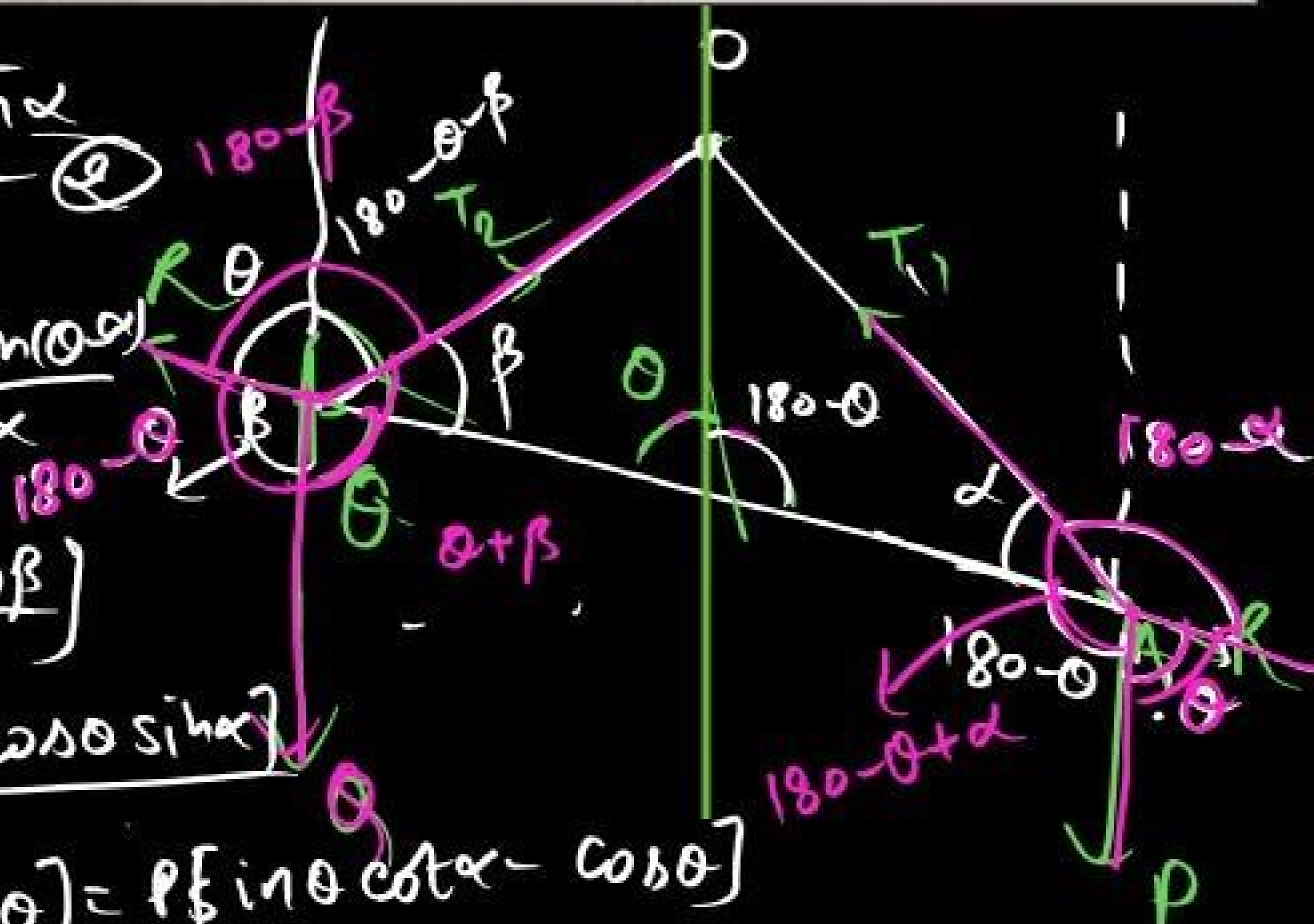
$$\Rightarrow Q\left[\frac{\sin\theta\cos\beta+\cos\theta\sin\beta}{\sin\beta}\right]$$

$$= P\left[\frac{\sin\theta\cos\alpha-\cos\theta\sin\alpha}{\sin\alpha}\right]$$

$$\Rightarrow Q[\sin\theta\cot\beta+\cos\theta] = P[\sin\theta\cot\alpha-\cos\theta]$$

$$\Rightarrow \sin\theta[-Q\cot\beta+P\cot\alpha] = (P+Q)\cos\theta$$

$$\Rightarrow \tan\theta = (P+Q)/P\cot\alpha - Q\cot\beta.$$



1.18. EQUILIBRIUM OF BODIES PLACED ON A SMOOTH INCLINED PLANE

(a) A body of weight W is placed on a smooth inclined plane of inclination α and is supported by a force acting horizontally. To find the force and the reaction of the plane.

The plane being smooth, the normal reaction R of the plane on the body placed at O is along the perpendicular to the inclined plane of inclination α. Let P be the horizontal force supporting the body.

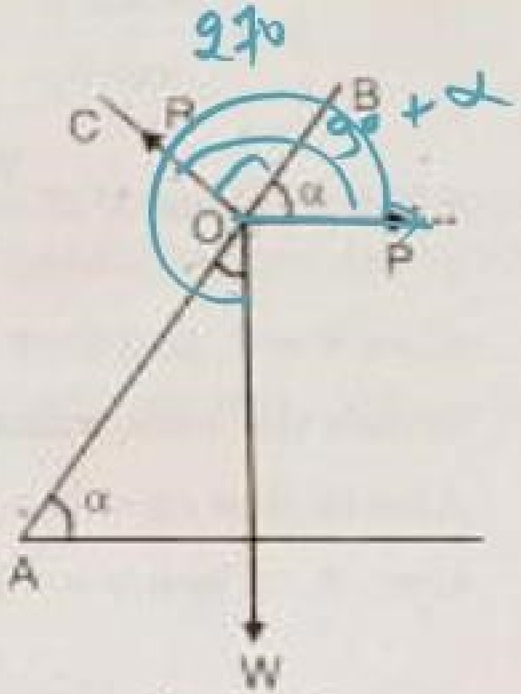


Fig. 1.45

Now, the body is in equilibrium under the action of the following forces acting at O :

(*i*) W, the weight of the body, acting vertically downwards

(*ii*) Force P acting horizontally

(*iii*) Normal reaction R along OC.

Resolving the forces horizontally and vertically, we have

$$P + R\cos(90° + \alpha) = 0$$

i.e.,
$$P - R\sin\alpha = 0$$

∴
$$P = R\sin\alpha$$

and
$$R\sin(90° + \alpha) - W = 0$$

∴
$$R\cos\alpha = W$$

or
$$R = \frac{W}{\cos\alpha} = W\sec\alpha \quad ...(1)$$

∴
$$P = \frac{W}{\cos\alpha}(\sin\alpha) = W\tan\alpha \quad ...(2)$$

$$\frac{W}{\sin(90+\alpha)} = \frac{R}{\sin 90°} = \frac{P}{\sin(180-\alpha)}$$

$$\frac{W}{\cos\alpha} = \boxed{R = \frac{P}{\sin\alpha}}$$

$$P\cos 0° + R\cos(90+\alpha) + W\cos 270 = 0$$

$$P\sin 0° + R\sin(90+\alpha) + W\sin 270 = 0$$

$$R\cos\alpha - W = 0$$

(*b*) ***A body of weight W is placed on a smooth inclined plane of inclination α and is kept in equilibrium by a force P which acts in a vertical plane in a direction making an angle θ with the plane i.e., with the line of greatest slope through the body. To find the magnitude of P and the normal reaction.***

(along the direction of plane.)

Let the force P act along OC making an angle θ with the line of greatest slope. The plane being smooth, the reaction R of the plane of the body at O is along the normal to the inclined plane.

Now the following three forces acting at O are in equilibrium :

(*i*) The force P along OC.

(*ii*) Normal reaction R along OD

(*iii*) W, the weight of the body, acting vertically downwards.

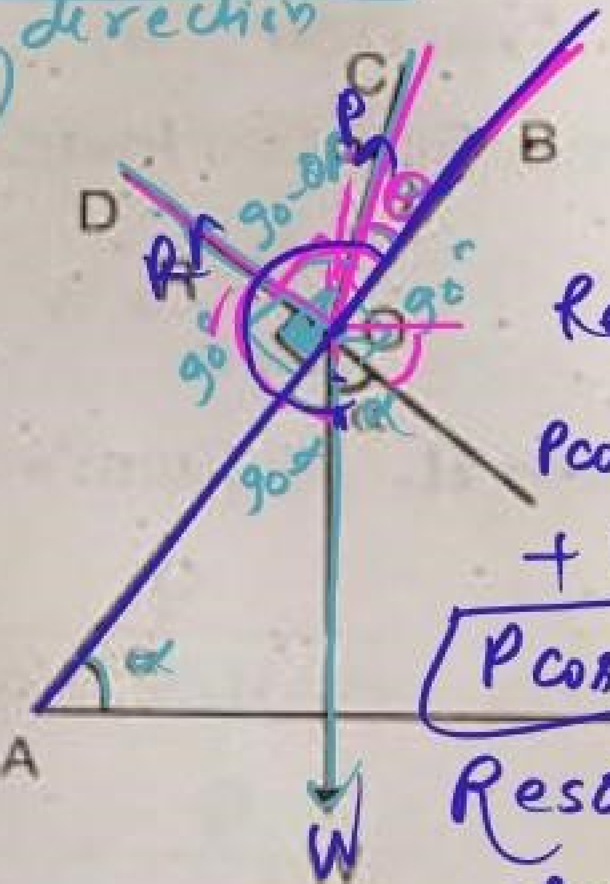


Fig. 1.46

Resolving along AB

$P\cos\theta + R\cos 90^\circ + W\cos(270-\alpha) = 0$

$\boxed{P\cos\theta = W\sin\alpha}$

Resolving $\perp$r to the plane

$P\sin\theta + R\sin 90^\circ + W\sin(270-\alpha) = 0$

$\boxed{P\sin\theta + R - W\cos\alpha = 0}$

Hence, by Lami's theorem at O, we have

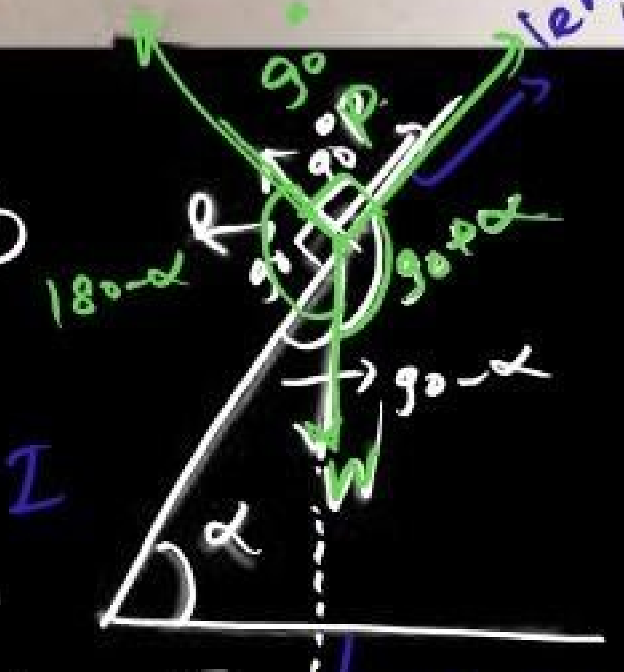
$$\frac{P}{\sin(180^\circ - \alpha)} = \frac{R}{\sin(90^\circ + \alpha + \theta)} = \frac{W}{\sin(90^\circ - \theta)}$$

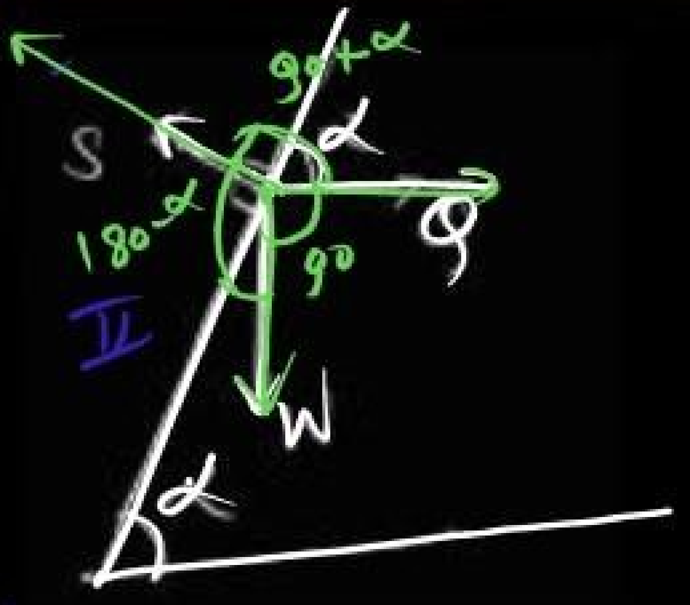
$$\frac{P}{\sin\alpha} = \frac{R}{\cos(\alpha + \theta)} = \frac{W}{\cos\theta}$$

$$\therefore \quad P = \frac{W\sin\alpha}{\cos\theta} \quad \text{and} \quad R = \frac{W\cos(\alpha + \theta)}{\cos\theta}$$

Example 1. *Two forces P and Q acting parallel to the length and the base of a smooth inclined plane would, each of them, singly support a weight W on the plane. Prove that*

$$\frac{1}{P^2} - \frac{1}{Q^2} = \frac{1}{W^2}.$$





Let α be the inclination of plane

In diagram I, by Lami's theorem

$$\frac{R}{\sin(90+\alpha)} = \frac{P}{\sin(180-\alpha)} = \frac{W}{\sin 90^\circ}$$

$$\Rightarrow \frac{R}{\cos\alpha} = \frac{P}{\sin\alpha} = W \quad \text{——} (1)$$

In diagram II, by Lami's theorem

$$\frac{Q}{\sin(180-\alpha)} = \frac{S}{\sin 90} = \frac{W}{\sin(90+\alpha)}$$

$$\frac{Q}{\sin\alpha} = \frac{S}{1} = \frac{W}{\cos\alpha} \quad \text{——} (2)$$

from (1)

$$\frac{W}{P} = \operatorname{cosec}\alpha \quad \text{——} (3)$$

from (2)

$$\frac{W}{Q} = \cot \quad \text{——} (4)$$

Now we know

$$\operatorname{cosec}^2\alpha - \cot^2\alpha = 1 \quad \text{——} (5)$$

Using (3) & (4) in (5)

$$\frac{W^2}{P^2} - \frac{W}{Q^2} = 1 \Rightarrow \frac{1}{P^2} - \frac{1}{Q^2} = \frac{1}{W^2}$$

Two weights P and Q rest on each of two smooth planes placed back to back, of inclination α and β, being connected by a string which runs horizontally from one to the other. Show that $P \tan \alpha = Q \tan \beta$.

If the string passes over a smooth pulley at the top of inclined planes, show that $P \sin \alpha = Q \sin \beta$.

C.W.

At A

$$\frac{R}{\sin 90^\circ} = \frac{T}{\sin(180-\alpha)} = \frac{P}{\sin(90+\alpha)}$$

$$\frac{R}{1} = \frac{T}{\sin\alpha} = \frac{P}{\cos\alpha}.$$

$$T = P\tan\alpha$$

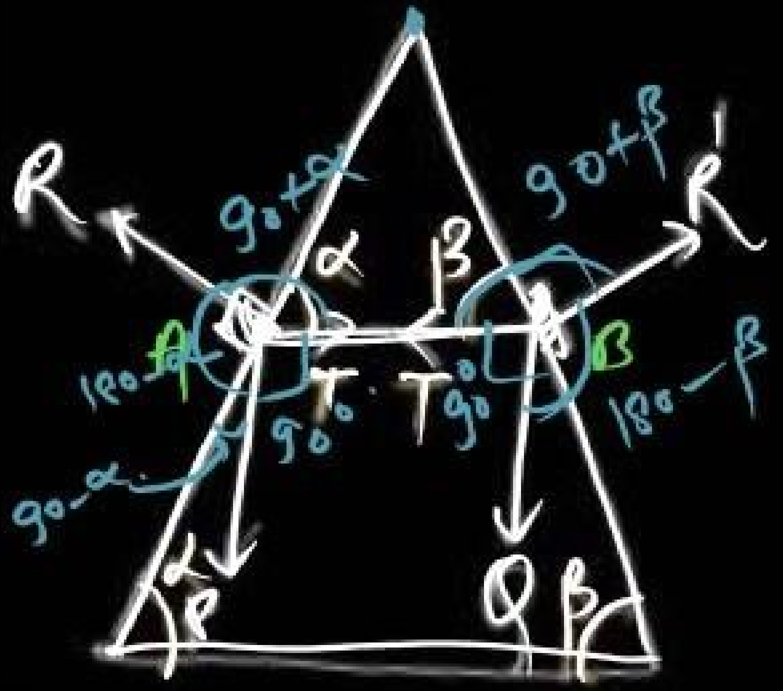
$$T = Q\tan\beta$$

$$P\tan\alpha = Q\tan\beta$$

At B

$$\frac{Q}{\sin(90+\beta)} = \frac{R'}{\sin 90^\circ} = \frac{T}{\sin(180-\beta)}$$

$$\frac{Q}{\cos\beta} = R' = \frac{T}{\sin\beta}$$



At A; by Lami's theorem;

$$\frac{P}{\sin 90} = \frac{T'}{\sin(180-\alpha)} = \frac{R}{\sin(90+\alpha)}$$

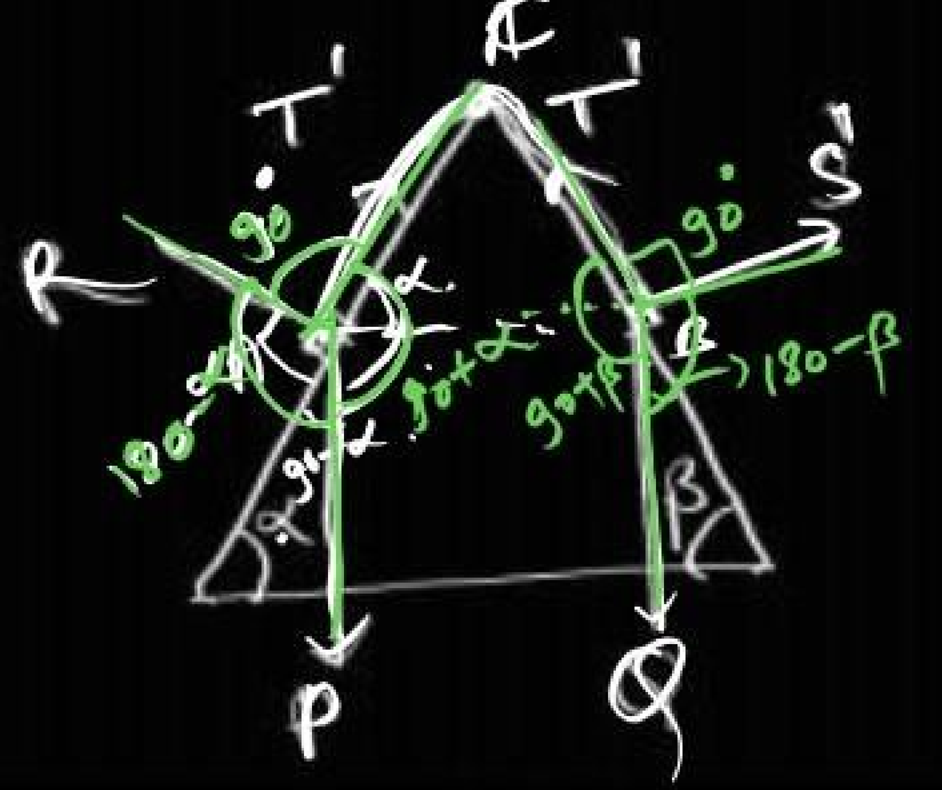
$$P = \frac{T'}{\sin\alpha} = \frac{R}{\cos\alpha}. \Rightarrow T' = P\sin\alpha \quad \text{--- (3)}$$

At B;

$$\frac{Q}{\sin 90^\circ} = \frac{S'}{\sin(90+\beta)} = \frac{T'}{\sin(180-\beta)}$$

$$\Rightarrow T' = Q\sin\beta \quad \text{--- (4)}$$

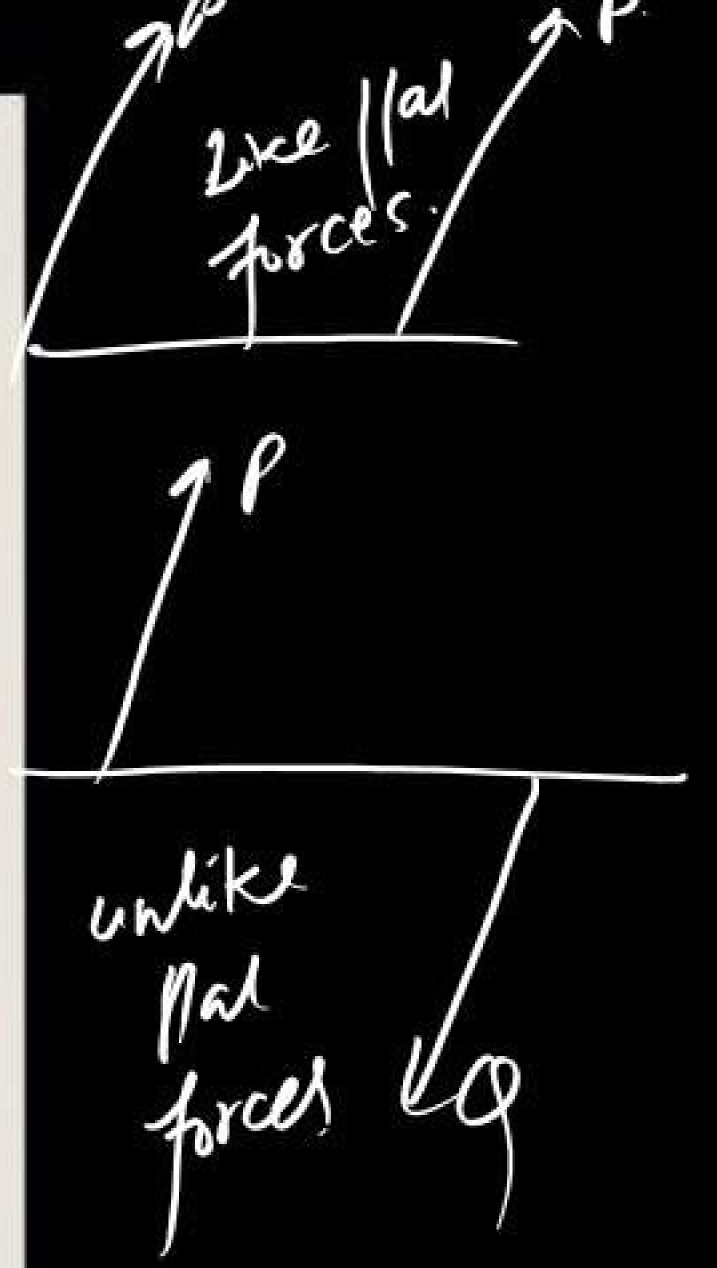
$$(3) = (4) \Rightarrow P\sin\alpha = Q\sin\beta.$$



PARALLEL FORCES

INTRODUCTION

Uptill now we have studied about two or more forces acting at a point. The resultant of these forces can be determined by using parallelogram law of forces, (if the forces are two in number) or by resolving the forces along two mutually perpendicular directions in the plane of the forces, (if the number of forces acting at a point in a plane is more than two). In case two forces act at two distinct points of a rigid body, then they are supposed to act on a particle placed at a point where the lines of action of the forces intersect provided these lines are coplanar and not parallel. The resultant of such forces can be determined by using parallelogram law of forces. Now we shall discuss the *forces which act on a rigid body having their lines of action parallel to each other*. These types of forces are called **parallel forces**. The resultant of these forces cannot be found by parallelogram law of forces since the line of action of these forces do not meet at a point. We shall now discuss the method of finding the resultant of parallel forces acting on a rigid body.



LIKE AND UNLIKE PARALLEL FORCES

Definition. *Two parallel forces are said to be **like** when they act in the same direction and **unlike** when they act in the opposite directions.*

Let P and Q (P > Q) be two like parallel forces acting at point A and B of a rigid body.

Hence the resultant of P and Q is $\boldsymbol{P + Q}$*, acting along OC at C parallel to the original direction of forces P and Q.*

To find the position of C :

Since Δ's OCA and AEH are similar

$$\therefore \quad \frac{AC}{OC} = \frac{AE}{EH} = \frac{S}{P} \qquad \ldots(1)$$

Again since Δ's OCB and BFG are similar

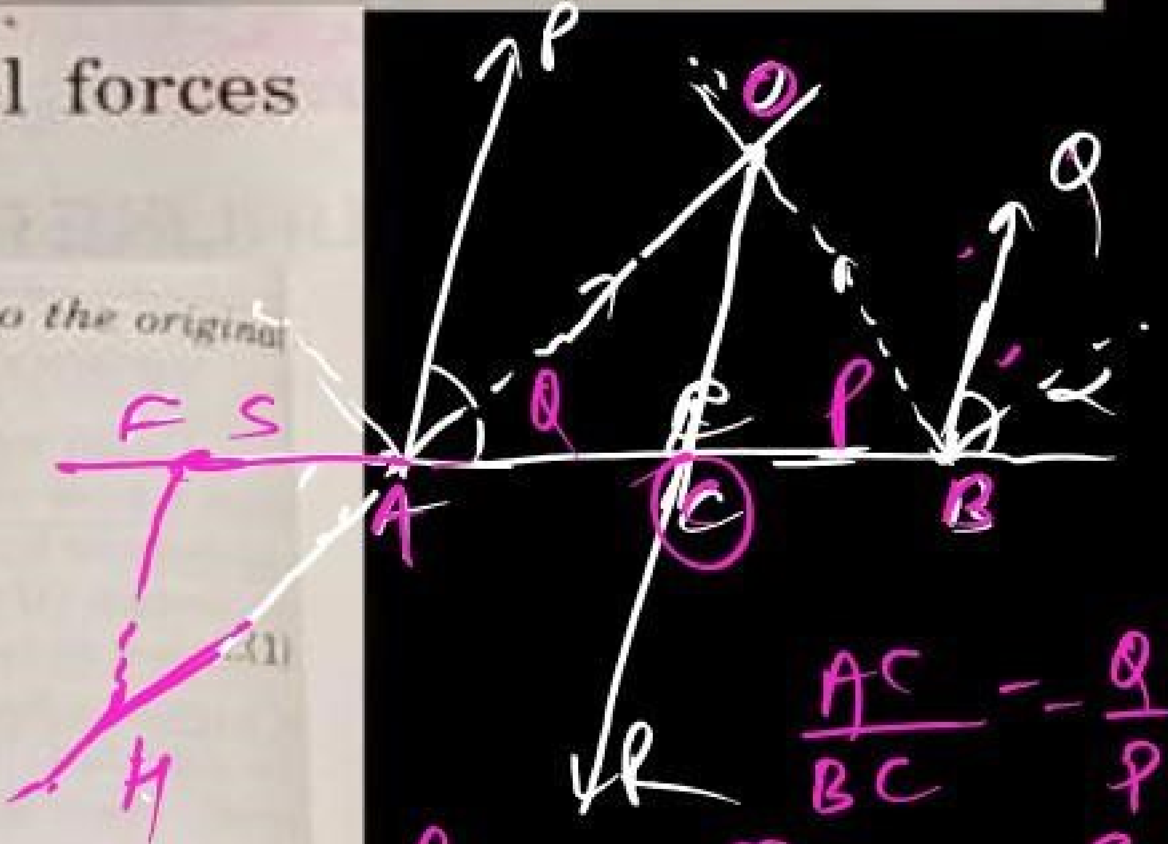
$$\therefore \quad \frac{CB}{OC} = \frac{BF}{FG} = \frac{S}{Q} \qquad \ldots(2)$$

Dividing (1) by (2), we have

$$\frac{AC}{CB} = \frac{Q}{P}$$

or $$\boxed{P \cdot AC = Q \cdot CB}$$

Hence the resultant is parallel to the given forces and divides AB internally in the ratio Q : P, i.e., in the inverse ratio of the forces.



$$\frac{AC}{BC} = \frac{Q}{P}$$

$$\frac{P}{BC} = \frac{Q}{AC} = \frac{R}{AB}$$

RESULTANT OF TWO UNEQUAL UNLIKE PARALLEL FORCES ACTING ON A RIGID BODY

Let P and Q (P > Q) be two unlike parallel forces acting at points A and B of a rigid body.

Hence the resultant of P and Q is **P – Q** acting along OC *i.e.*, acting at C in the direction of greater force P.

To find the position of C :

Since Δ's ADF and ACO are similar,

$$\therefore \quad \frac{AC}{CO} = \frac{AD}{DF} = \frac{S}{P} \qquad \ldots(1)$$

Again since Δ's BEG and BCO are similar,

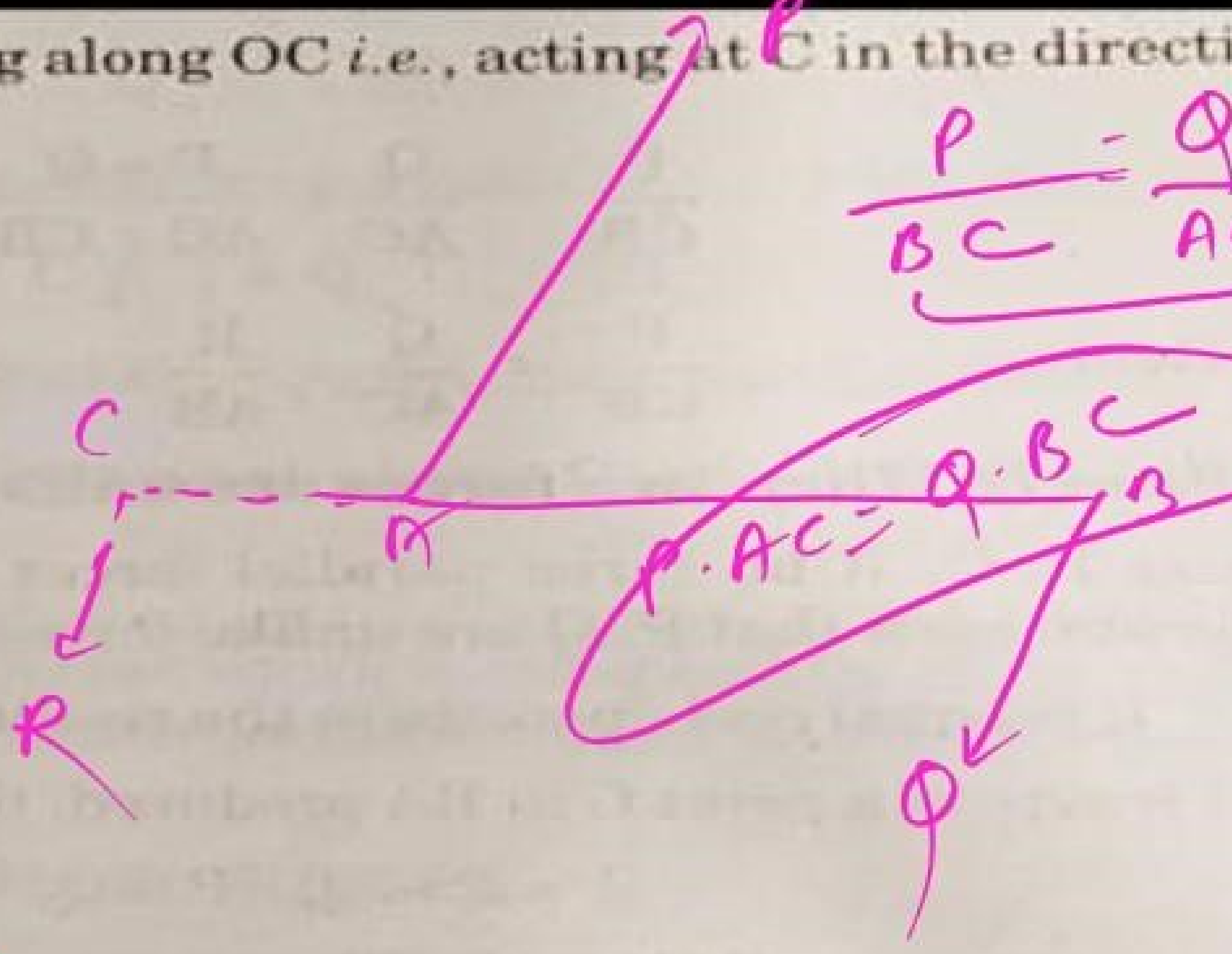
$$\therefore \quad \frac{BC}{CO} = \frac{BE}{EG} = \frac{S}{Q} \qquad \ldots(2)$$

Dividing (1) by (2), we have

$$\frac{AC}{BC} = \frac{Q}{P} \qquad \ldots(3)$$

or

$$\mathbf{P \cdot AC = Q \cdot BC}$$



$$\frac{P}{BC} = \frac{Q}{AC} = \frac{R}{AB}$$

$$P \cdot AC = Q \cdot BC$$

Hence the resultant is parallel to the given forces in the same sense with the large force P and divides AB externally in the ratio Q : P i.e., in the inverse ratio of the forces.

2.5. ANALOGUE OF LAMI'S THEOREM

If three parallel forces acting on a rigid body are in equilibrium, ea proportional to the distance between the other two

Case I. When two forces are like :

Let P, Q, R be three parallel forces which are in equilibrium such that P and Q are like forces.

$\therefore$ R is equal and opposite to the resultant of P and Q

If R acts at a point C in AB, then R = P + Q, and

$$P \cdot AC = Q \cdot CB$$

$$\therefore \quad \frac{P}{CB} = \frac{Q}{AC}$$

or
$$\frac{P}{CB} = \frac{Q}{AC} = \frac{P+Q}{AC+CB}$$

Hence,
$$\frac{P}{CB} = \frac{Q}{AC} = \frac{R}{AB}.$$

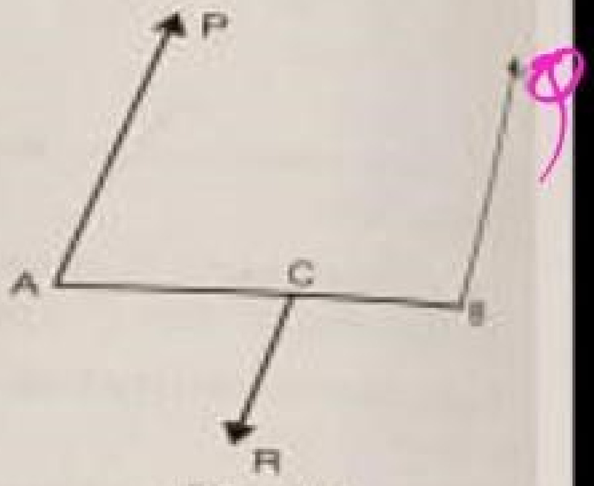


Fig. 2.3

Case II. When two forces are unlike :

Let P, Q, R be three parallel forces which are in equilibrium such that P, Q are unlike forces

$\therefore$ R is equal and opposite to the resultant of P and Q

If R acts at a point C in BA produced, then

$$R = P - Q, \; (P > Q)$$

and
$$P.CA = Q.CB$$

$$\therefore \quad \frac{P}{CB} = \frac{Q}{CA}$$

or
$$\frac{P}{CB} = \frac{Q}{CA} = \frac{P-Q}{CB-CA}$$

Hence,
$$\frac{P}{CB} = \frac{Q}{AC} = \frac{R}{AB}$$

Hence the theorem.

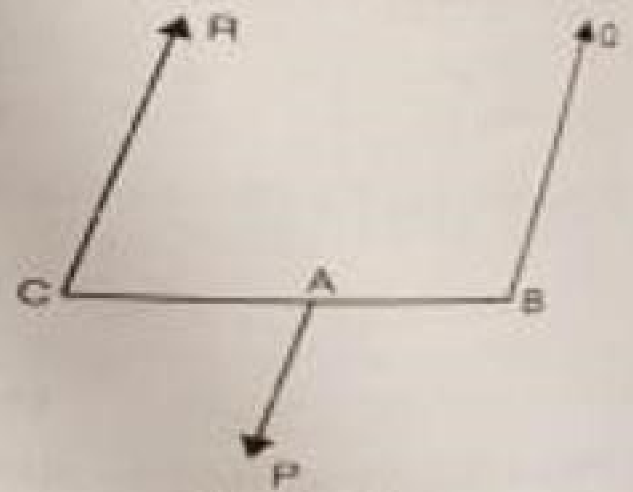


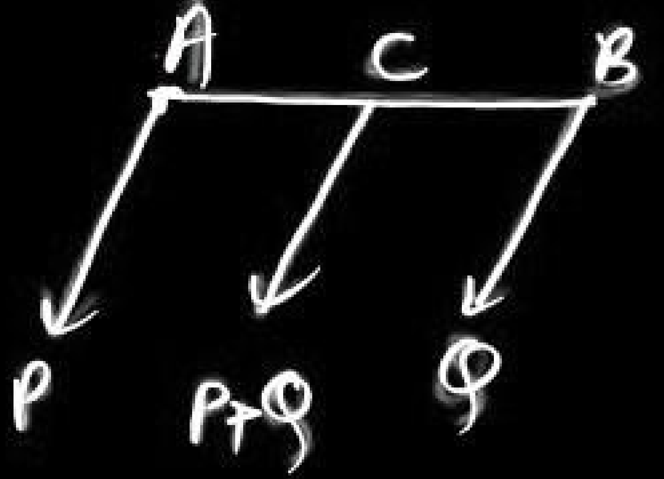
Fig. 2.4

Two like parallel forces P and Q (P > Q) act upon a rigid body at A and B respectively. Let P and Q be interchanged in position, show that the point of application of the resultant will be displaced through a distance x along AB given by

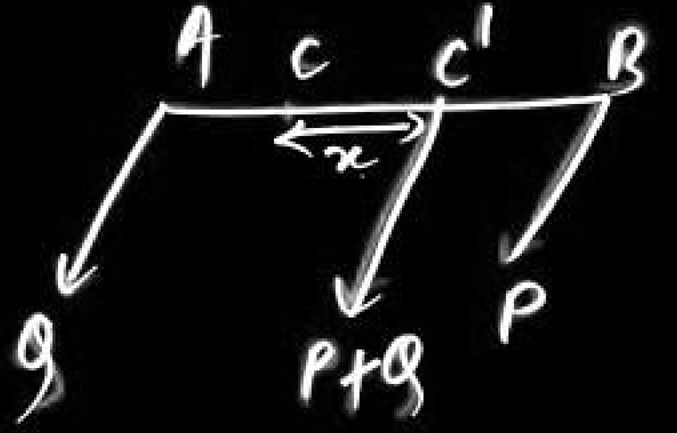
$$x = \frac{P - Q}{P + Q} AB \qquad \text{[K.U. 2001]}$$

A.T.Q.1 by Lami's Analogue theorem

$$\frac{P}{BC} = \boxed{\frac{Q}{AC} = \frac{P+Q}{AB}} \quad \text{— (1)}$$



dig. 1



dig 2

A.T.Q.2; $\frac{P}{AC'} = \frac{Q}{C'B} = \frac{P+Q}{AB}$

$\Rightarrow \frac{P}{AC+CC'} = \frac{Q}{C'B} = \frac{P+Q}{AB}$ — (2)

By (1) & (2) $\frac{Q}{AC} = \frac{P}{AC+x} \Rightarrow \frac{AC+x}{AC} = \frac{P}{Q}$

$\Rightarrow \frac{x}{AC} = \frac{P}{Q} - 1 \Rightarrow x = \frac{P-Q}{Q} AC$

Again By (1)

$AC = \frac{Q}{P+Q} AB$

$$\boxed{x = \frac{P-Q}{P+Q} AB}$$

Two unlike parallel forces P and Q $(P > Q)$, x metre apart act at two points of a rigid body. Show that if direction of P be reversed, the resultant is displaced through a distance $\frac{2PQ}{P^2 - Q^2}x$ metres.

H.W.

A.T.D. (1)

$$\frac{P}{BC} = \boxed{\frac{Q}{AC} = \frac{P-Q}{AB=x}} \quad \text{—(1)}$$

A.T.D. (2)

$$\frac{P}{BC'} = \boxed{\frac{Q}{AC'} = \frac{P+Q}{AB=x}}$$

By (1) $AC = \frac{Qx}{P-Q}$

$AC' = \frac{Qx}{P+Q}$

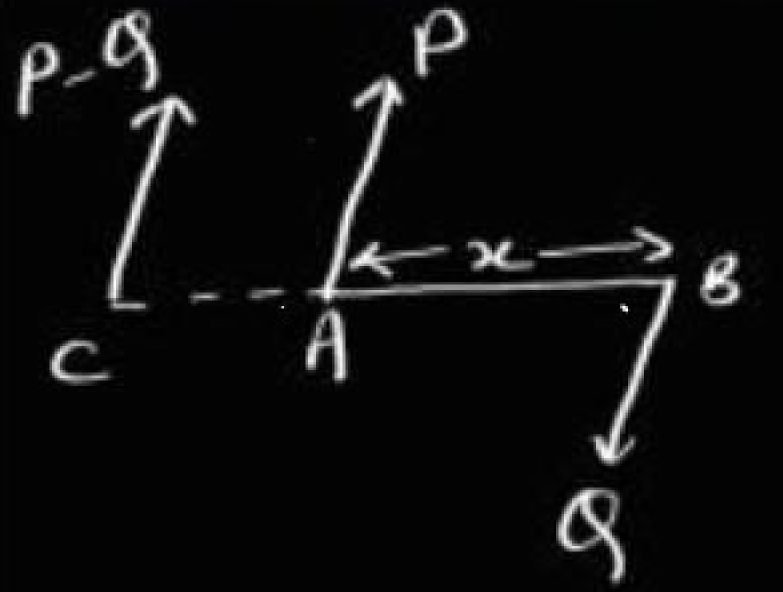
Required distance

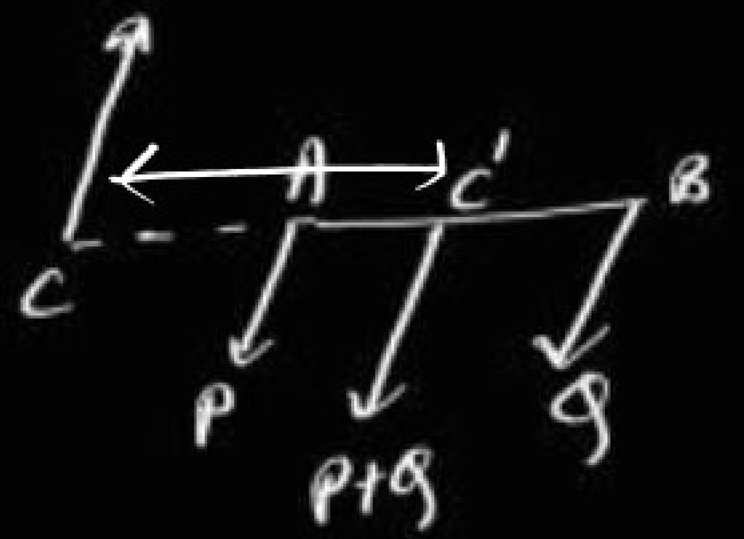
$$CC' = AC + AC'$$

$$= \frac{Qx}{P-Q} + \frac{Qx}{P+Q}$$

$$= \frac{Qx[P+\cancel{Q}+P-\cancel{Q}]}{(P-Q)(P+Q)}$$

$$= \frac{2PQx}{P^2-Q^2}$$

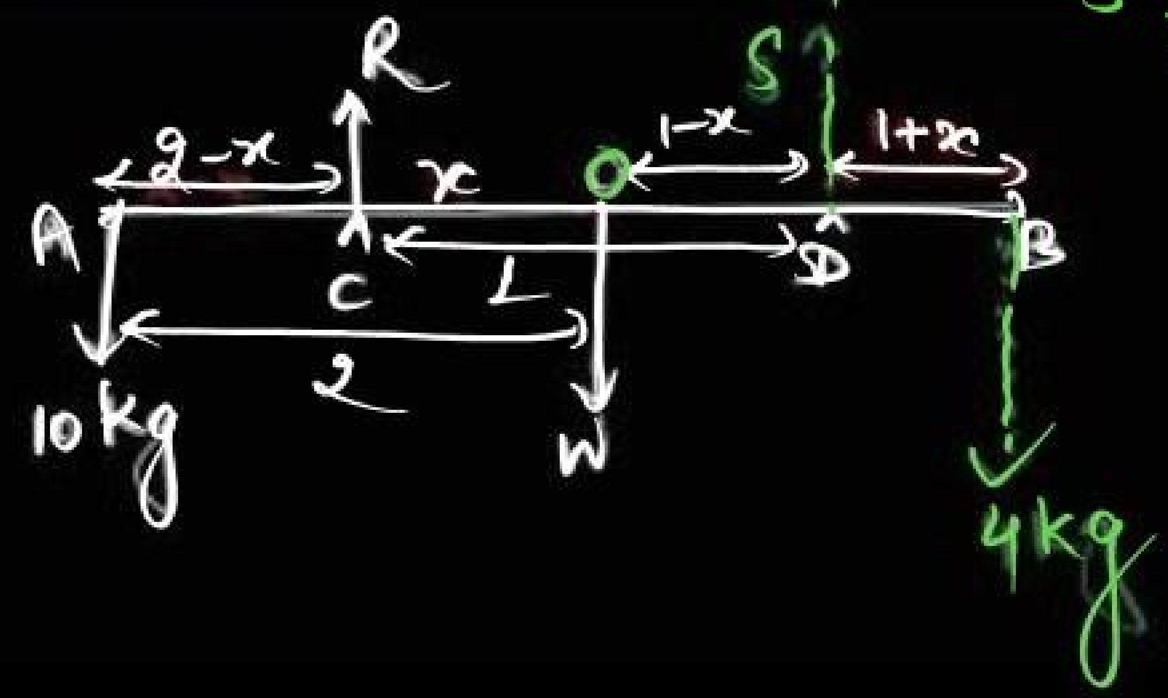




A heavy uniform rod 4 m long rest horizontally on two pegs which are 1 m apart. A weight of 10 kg suspended from one end or a weight of 4 kg suspended from the other end will just tilt the rod up. Find the weight of the rod and the distances of the pegs from the centre of the rod.

H.W.

Ans $W = 20$ kg
dist $\rightarrow \frac{2}{3}$ m & $\frac{1}{3}$ m



A uniform rod of length $2l$ and weight W is lying across two pegs on the same level d metre apart. If neither peg can stand a stress greater than T, show that the length of the rod which can project beyond either peg cannot be greater than

$$l - \frac{d(W-T)}{W}$$

H.W.

A.T. Lami's Analogue theorem

$$\frac{R_1}{OD = x} = \frac{R_2}{OC = d-x} = \frac{W}{CD = d}$$

$$\frac{R_1}{x} = \frac{W}{d}$$

$$\Rightarrow x = \frac{R_1 d}{W}$$

$$R_2 = \frac{W(d-x)}{d}$$

$$x = \frac{-R_2 d + Wd}{W}$$

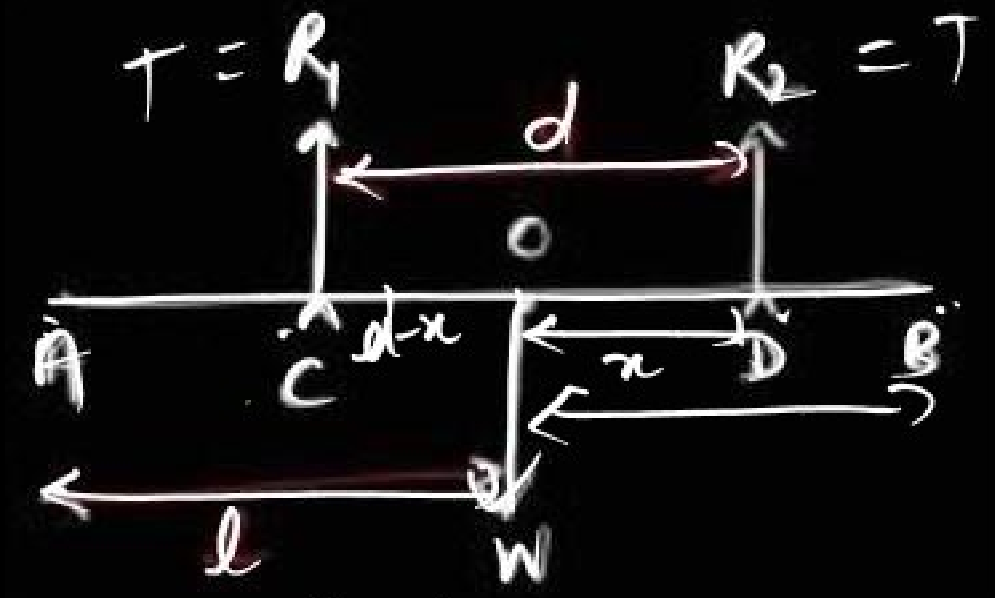
To find out $AC = l-(d-x)$

$$= l-d+x$$

$$= l-d+\frac{R_1 d}{W}$$

AC is maxi. if $R_1 = T \Rightarrow AC_{max} = l-d+\frac{Td}{W}$

$$= l - d\left(\frac{W-T}{W}\right)$$



$$BD = l - x$$

$$= l - \left(\frac{Wd - R_2 d}{W}\right)$$

$$BD_{max} = l - \left(\frac{Wd - Td}{W}\right)$$

$$= l - d\left(\frac{W-T}{W}\right)$$

Three like parallel forces P, Q, R act at the corners of a triangle ABC. Prove that their centre is

(i) *the centroid of the triangle if* $P = Q = R$

(ii) *the orthocentre of the triangle if* $\dfrac{P}{\tan A} = \dfrac{Q}{\tan B} = \dfrac{R}{\tan C}$ [K.U. 1995]

(i) centroid

Let $P = Q = R$

Resultent of R & Q is at D it is $R+Q$ i.e. $2P$ which is mid-point of BC

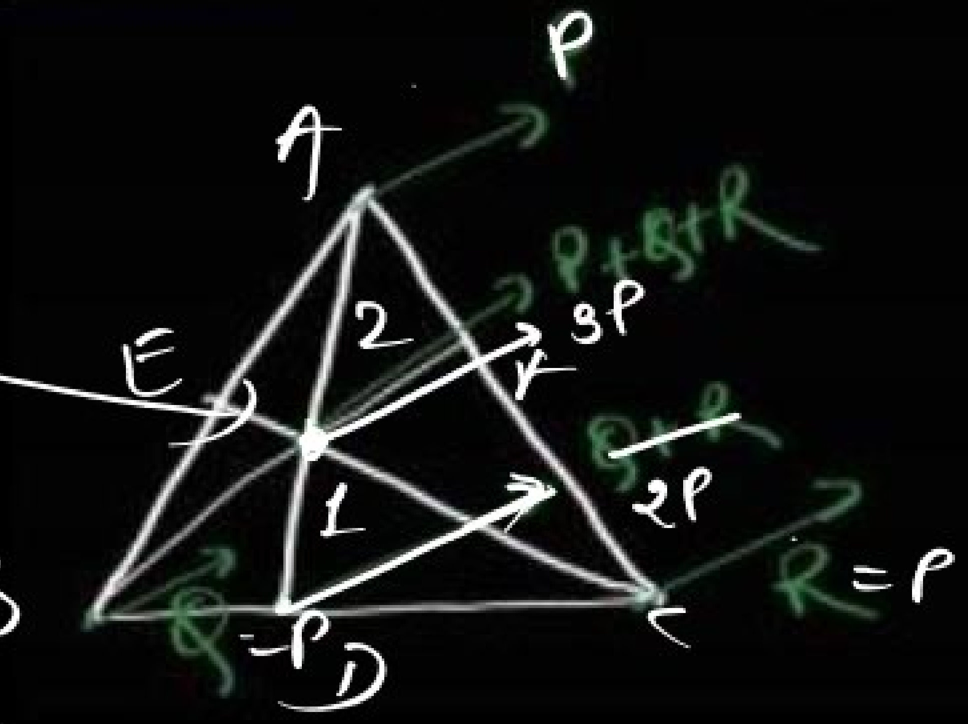
$\therefore$ At D; we have force $2P$ & it is $\parallel$al to P.

$\therefore$ Resultant of P & $2P$ would divide AD into $2:1$

Let that point is G. Hence G divides median into $2:1$

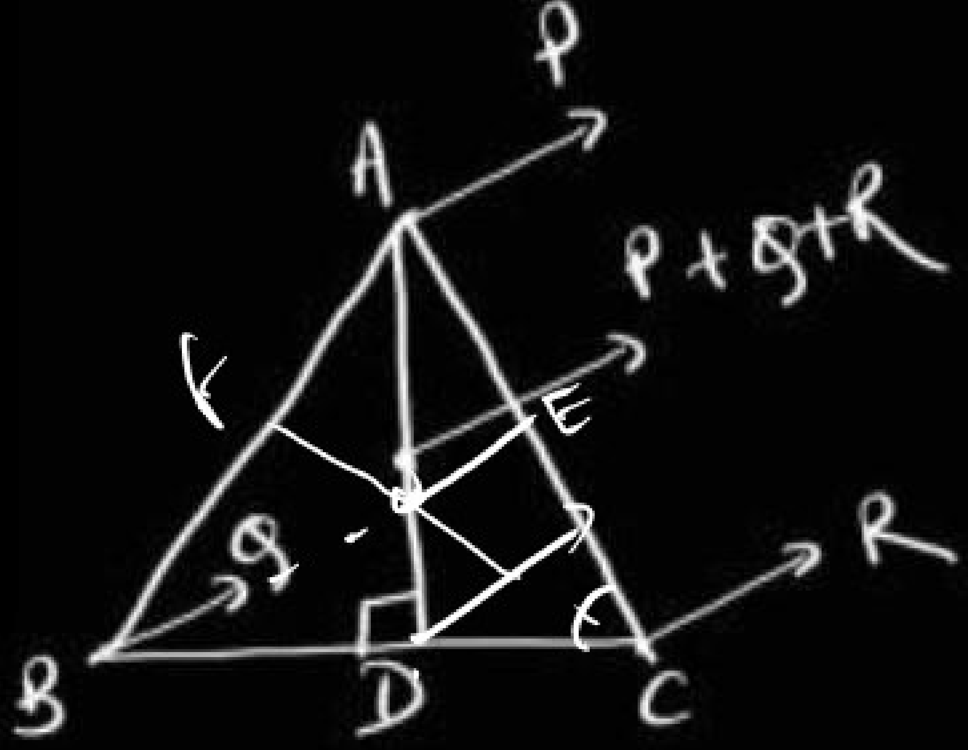
$\rightarrow G$ is centroid

centroid



(ii)- orthocentre

Given is $\frac{P}{\tan A} = \boxed{\frac{Q}{\tan B} = \frac{R}{\tan C}}$ — (1)



$$\underbrace{\frac{Q}{DC} = \frac{R}{BD}} = \frac{Q+R}{BC}$$

$$\boxed{\frac{Q}{R} = \frac{DC}{BD}}$$

From (1) $\frac{Q}{R} = \frac{\tan B}{\tan C}$.

$\frac{DC}{AD} = \cot C$

$$\frac{DC}{BD} = \frac{\tan B}{\tan C}$$

$$\frac{DC}{BD} = \frac{AD\cot C}{AD\cot B} \Rightarrow DC = AD\cot C.$$

$\tan C = \frac{AD}{DC}$.

$\Rightarrow B =$ Right angle.