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Test- Abstract Algebra- Groups (Evening Batch)

Batch: Evening batch

Date: 10-Feb-2024

Mains Answer Writing Guidance Programme	
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Medium	English
Date	11-02-2024
Subject and Test Number	Mathematics

Instructions:
1. Please scan your answers and form single pdf and share within 48 hours.
2. Writing in the margins leads to rejection of copy.
3. Kindly take due appointment with coordination team to discuss the answer copy with respective mentor.
4. Copies will be evaluated within 7 days of submission.

Evaluated	Reviewed

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Criteria/Parameters	Exellent	Very Good	Good	Average	Poor
Language and Articulation					
Content and Conceptual Clarity					
Number of Questions Attempted					
Structure and Presentation					
Coherence and Structuration					

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Examiner's Feedback

Q. No.

All the questions are compulsory and carry equal marks.

1 If in the group G, $a^5 = e$, $aba^{-1} = b^2$ for some $a, b \in G$, find the order of b.

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Given:- G is a group
$a^5 = e \Rightarrow o(a) \le 5$
$aba^{-1} = b^2$, for some $a, b \in G$

To find:- Order of b

Since $a \in G$
$\Rightarrow a^{-1} \in G$

Consider $a^5 = e$
post-multiply by 'a^{-1}'
$a^5 . a^{-1} = a^{-1}$
$a^4 = a^{-1}$
post-multiply by (a^{-1})
$a^4 . a^{-1} = a^{-1} . a^{-1}$
$a^3 = (a^{-1})^2$

Now consider,
$aba^{-1} = b^2$
$a^2ba^{-1} = ab^2$
$a^3ba^{-1} = a^2b^2$

$aba^{-1} = b^2$
$ab = b^2a$
$ab = b . ba$
$(a^{-1})^2 b$

$aba^{-1} = b^2$
x by a^{-1}
$a^{-1} . aba^{-1} = a^{-1}b$
$ba^{-1} = a^{-1}b$
$ba^4 = a^4b$

$b^m = e$
x by a
$ab^m = a$
x by a^{-1}
$ab^ma^{-1} = a . a^{-1}$
$ab^ma^{-1} = e$
$ab^1a^{-1} = b^2$

$aba^{-1} = b^2$
$(aba^{-1})^2 = (aba^{-1})(aba^{-1})$
$= abba^{-1}$
$= ab^2a^{-1}$

$ab^2a^{-1} = b^4$

$aba^{-1} = e$

$aba^{-1} = b^2$
$b^2 = e$
$o(b) = 2$

$b^{2m} = e$
$o(b) = 2m$
$= 2 \times 2$
$= 4$

$a^5 = e$ $aba^{-1} = b^2$
$ba^4 = a^4b$ ba

Procedure

1st find b^4 then b^8 b^{16} b^{32} then $b = b$
$\Rightarrow b^{31} = e.$
$o(b) = 31$

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Given:- 'G' is a group and

$a^5 = e$, $\boxed{aba^{-1} = b^2} \rightarrow (1)$

Let 'm' be the order of 'b'.

$\Rightarrow \quad O(b) = m$

$\Rightarrow \quad b^m = e$.

pre-multiply by 'a' and post-multiply by 'a^{-1}',

$ab^m a^{-1} = a \cdot e \cdot a^{-1}$

$ab^m a^{-1} = e$

$\boxed{ab^m a^{-1} = b^{2m}} \rightarrow (2)$

comparing equation (1) and (2),

$b^m = b'$

$\boxed{m = 1}$

$\Rightarrow b' = e$

$\Rightarrow b^2 = e$

this can not be compared as here 1-1 is not given. two different element also can give you same result as well.

$\therefore$ $\boxed{'1'}$ is the order of 'b'.

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2 Let $G = IR - \{-1\}$ be the set of all real numbers omitting -1. Define the binary relation $*$ on G by $a*b = a+b+ab$. Show $(G,*)$ is a group and it is abelian.

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Given:-

$G = R - \{-1\}$

'$*$' defined by $a*b = a+b+ab$

To show $(G,*)$ is a group:-

A group has to satisfy 4 properties.

1) Closure :- Let $a, b \in G$, then

$a*b = a+b+ab \in G$

$\therefore G$ is closed under '$*$'.

2) Associativity:- Let $a, b, c \in G$,

Consider $a*(b*c) = a*(b+c+bc)$

$= a+b+c+bc+a(b+c+bc)$

$= a+b+c+bc+ab+ac+abc$

Again consider $(a*b)*c = (a+b+ab)*c$

$= a+b+ab+c+(a+b+ab)c$

$= a+b+ab+c+ac+bc+abc$

$= a*(b*c)$

$\Rightarrow (a*b)*c = a*(b*c)$

$\therefore G$ is associative under $*$.

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3) Existence of Identity :-

Let $e \in G$ such that $a*e = a = e*a$

$a*e = a$

$a + e + ae = a$

$e + ae = 0$

$e(1+a) = 0$

$\boxed{e = \frac{1}{1+a}} \in G$, for $a \neq -1$ $\Rightarrow$ Either $\boxed{e = 0}$ or $1+a = 0$ $\Rightarrow a = -1$

$\hookrightarrow$ identity, not possible as $-1 \notin R$.

$\therefore$ Identity of a Group exists in 'G'.

4) Existence of Inverse :-

Let $a, b \in G$ such that $a*b = e = b*a$

$a*b = e$

$a*b = \frac{1}{1+a}$ $\Rightarrow$ $a*b = 0$

$a + b + ab = \frac{1}{1+a}$ $\quad$ $a+b+ab = 0$

$b(1+a) = \frac{1}{1+a} - a$ $\quad$ $b(1+a) = -a$

$\Rightarrow b = \frac{-a}{1+a}$

$b = \frac{1}{1+a}\left[\frac{1}{1+a} - a\right] = \frac{1 - \cancel{a} - a^2}{(1+a)^2}$

$\boxed{b = \frac{1-a-a^2}{(1+a)^2}} \in G$ for $a \neq -1$, where $\boxed{b = a^{-1}}$

$\therefore$ Inverse of a Group exists in 'G'.

Since $(G, *)$ satisfies all 4 properties, it is a Group.

To show Abelian :- Let $a, b \in G$ such that $a*b = b*a$

$a*b = a + b + ab = b + a + ab = b + a + ba$

$= b*a$

$\therefore \boxed{a*b = b*a} \rightarrow$ commutativity

$\therefore (G, *)$ is ABELIAN

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3 Let p be a prime number and Z_p denote the additive group of integers modulo p. Show that every non-zero element of Z_p generates Z_p.

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Given:-

$p \to$ prime number

$Z_p = \{ a +_p b : a, b \in Z, p \text{ is prime} \}$

$Z_p : \{0, 1, 2, 3, \ldots, p-1\}$

To show:- Every non-zero element of Z_p generates Z_p.

Let $x = a +_p b$

Consider $(Z_5, +_5) = \{\underbrace{1, 2, 3, 4}_{\text{non-zero elements}}\}$

$5 \to$ prime number

$1^1 = 1$

$1^2 = 1 +_5 1 = 2$

$1^3 = 1^2 +_5 1 = 2 +_5 1 = 3$

$1^4 = 1^3 +_5 1 = 3 +_5 1 = 4$

$\therefore$ 1 generates $\{1, 2, 3, 4\} = (Z_5, +_5)$

$2^1 = 2$

$2^2 = 2 +_5 2 = 4$

$2^3 = 2^2 +_5 2 = 4 +_5 2 = 1$

$2^4 = 2^3 +_5 2 = 1 +_5 2 = 3$

$\therefore$ 2 generates $\{1, 2, 3, 4\} = (Z_5, +_5)$

This is just an example

To prove a result we need to follow general laws

Let $a \in Z_p$ such that $a \neq 0$ then show that $o(a) = p$.

This way we use the result $\to o(\text{Generator}) = o(\text{Group})$

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$3^1 = 3$

$3^2 = 3 +_5 3 = 1$

$3^3 = 3^2 +_5 3 = 1 +_5 3 = 4$

$3^4 = 3^3 +_5 3 = 4 +_5 3 = 2$

$\therefore$ 3 generates $\{1,2,3,4\} = (Z_5, +_5)$

$4^1 = 4$

$4^2 = 4 +_5 4 = 3$

$4^3 = 4^2 +_5 4 = 3 +_5 4 = 2$

$4^4 = 4^3 +_5 4 = 2 +_5 4 = 1$

$\therefore$ 4 generates $\{1,2,3,4\} = (Z_5, +_5)$

The above example shows that every non-zero element of a Z_5 generates Z_5.

Hence proved.

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Q. No. 4

Prove that a group of Prime order is abelian.

Given:- (G, *) is a group of prime order

To show:- (G, *) is Abelian.

Let 'p' be the prime ~~number~~, and let $a \in G$.

Group of prime order $\Rightarrow O(G) = p$

For $p = 2$:-

Let $G = \{e, a\}$

To show 'G' is Abelian:-

Since $e \in G \Rightarrow e^{-1} \in G$

$a \in G \Rightarrow a^{-1} \in G$.

$ae = e = ea$

$\therefore ae = ea \Rightarrow$ 'G' is Abelian

For $p = 3$:-

Let $G = \{e, a, b\}$

Let $a^{-1} = b$ and $b^{-1} = a$, $e^{-1} = e$.

~~consider $ab = ba$~~

·	e	a	b
e	e	a	b
a	a	a^2	ab
b	b	ba	b^2

consider $\boxed{ab = e} \rightarrow b = a^{-1}$ ✓

But if $ab = a \Rightarrow b = e$ ✗

consider $\boxed{ba = e} \Rightarrow a = b^{-1}$ ✓

⇓

·	e	a	b
e	e	a	b
a	a	b	e
b	b	e	a

$\Rightarrow a^2 = b$ and $b^2 = a$

Since $ab = ba$

$\Rightarrow$ 'a' is Abelian.

Here 1st prove that Every group of prime order is cyclic & then prove that every cyclic group is abelian.

To prove a result examples are not sufficient. Examples are used to negate (counter) something.

Both the above results are in notes.

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For $p=5$:-

Let $G = \{e, a, b, c, d\}$

$e^{-1} = e.$

Let $a^{-1} = b$, $b^{-1} = a$, $c^{-1} = d$, $d^{-1} = c$

	e	a	b	c	d
e	e	a	b	c	d
a	a	a^2	ab	ac	ad
b	b	ba	b^2	bc	bd
c	c	ca	cb	c^2	cd
d	d	da	db	dc	d^2

Let $ab = e$
$\Rightarrow b = a^{-1}$ ✓

Again let $ba = e$
$\Rightarrow a = b^{-1}$ ✓

$\therefore \boxed{ab = ba}$

Let $ac = e \Rightarrow c = a^{-1}$ ✗
$\boxed{ac = b} \Rightarrow$
$ac = c \Rightarrow a = e$ ✗
$ac = a \Rightarrow c = e$ ✗
~~$ac = d \Rightarrow$~~

Let $ad = e \Rightarrow d = a^{-1}$ ✗
$ad = a \Rightarrow d = e$ ✗
$\boxed{ad = b} \Rightarrow$
~~$ad = c \Rightarrow$~~
$ad = d \Rightarrow a = e$ ✗

Let $bc = e \Rightarrow c = b^{-1}$ ✗
$bc = b \Rightarrow c = e$ ✗
$\boxed{bc = d} \Rightarrow$
$bc = c \Rightarrow b = e$ ✗
$bc = a \Rightarrow$

Let $bd = e \Rightarrow d = b^{-1}$ ✗
$\boxed{bd = a}$
$bd = b \Rightarrow d = e$ ✗
$bd = c$
$bd = d \Rightarrow b = e$ ✗

Let $cd = e \Rightarrow d = c^{-1}$ ✓
Again let $dc = e \Rightarrow c = d^{-1}$
$\therefore \boxed{cd = dc}$ $\Rightarrow$ 'G' is Abelian

$ac = b$
$ad \neq c$
$a^2 = d$
$b^2 = c$
$bc = a$ (e)
b
$bd = a$ (e)
b

	e	a	b	c	d
e	e	a	b	c	d
a	a	d	e	b	c
b	b	e	c	d	a
c	c	b	d	a	e
d	d	c	a	e	b

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consider,

$$a^{32} = a^8 . a^8 . a^8 . a^8$$
$$= e . e . e . e$$
$$a^{32} = e$$
$$\Rightarrow G = \langle a^{32} \rangle$$

⋮

$$a^{8n} = \underbrace{a^8 . a^8 . a^8 - - . a^8}_{\text{'n' times}}$$
$$a^{8n} = e$$
$$\Rightarrow G = \langle a^{8n} \rangle$$

From the above evaluation, we get to know that there are '$\boxed{n}$' generators of a cyclic group of order 8.

$\therefore$ $\boxed{\text{Number of generators} = n}$

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What are the order of the following permutations in S_{10}?

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 1 & 8 & 7 & 3 & 10 & 5 & 4 & 2 & 6 & 9 \end{pmatrix}$$ and (1 2 3 4 5) (6 7).

Given:-

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 1 & 8 & 7 & 3 & 10 & 5 & 4 & 2 & 6 & 9 \end{bmatrix}$$

(1) (2 8) (3 7 4) (5 10 9 6)

↳ 2 × 3 × 4 = 24

The permutations of Even order are :-

$\boxed{(1) \text{ and } (3\ 7\ 4)\ \}, \ (1\ 2\ 3\ 4\ 5)}$

The permutations of Odd order are:-

$\boxed{(2\ 8) \text{ and } (5\ 10\ 9\ 6)\ \}, \ (6\ 7)}$

Order of (1) = 1

Order of (2 8) = 2

Order of (3 7 4) = ~~4~~ 3

Order of (5 10 9 6) = ~~5~~ 4

Order of (1 2 3 4 5) = ~~6~~ 5

Order of (6, 7) = ~~3~~ 2

order of a cycle = no. of elements in the cycle.

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Order of a permutation

→ L.C.M of order of disjoint cycles.

In above case

Let element is
$(1\,2\,3\,4\,5)(6\,7)$
↳ $o(1\,2\,3\,4\,5) = 5$

& $o(6\,7) = 2$

$\therefore o[(1\,2\,3\,4\,5)(6\,7)] = LCM(5,2)$
$= 10$

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