

- REAL ANALYSIS -

PYQs

UPSC Mathematics optional (2025-16)

* Complete Written Solutions | 10 years |

Proof Based Approach *

• Compiled and solved by Surya Pratap Singh
Faculty, Plutus IAS.

• Examine whether the series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ is abs. or cond. convergent

Solⁿ First we check for $\sum_{n=1}^{\infty} \left| \frac{(-1)^{n-1}}{n} \right| = \sum_{n=1}^{\infty} \frac{1}{n}$ (Harmonic series)

To prove $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, we'll show it using the fact

If series is converges, then its ~~sum~~^{sequence} of partial sum is bounded.

So, We will ~~check only the~~^{show that} sequence of partial sum (SOPs) is unbounded.

$$\text{Consider, } S_{2n} = \sum_{i=1}^{2n} \frac{1}{i} = 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{4} + \dots + \frac{1}{2n}$$

$$S_{2n} = 1 + \frac{1}{2} + \left(\frac{1}{2} + \frac{1}{4} \right) + \left(\frac{1}{4} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} \right) + \dots + \left(\frac{1}{2^{n-1}+1} + \dots + \frac{1}{2^n} \right)$$

$$S_{2n} > 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4} \right) + \dots + \left(\frac{1}{2^n} + \frac{1}{2^n} + \dots + \frac{1}{2^n} \right)$$

$$S_{2n} > 1 + \frac{n}{2}$$

$\Rightarrow S_{2n}$ is unbounded, it implies $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

$\sum (-1)^n \frac{1}{n}$ converges, using Alternating Series Test -

• Alternating Series Test -

Let $\{x_n\}$ be $\downarrow$ sequ. & $x_n > 0 \forall n$ & $\lim(x_n) = 0$

Then Series $\sum (-1)^{n+1} x_n$ converges.

Here, $\sum (-1)^n \frac{1}{n} = \sum (-1)^n x_n$ where $x_n = \frac{1}{n}$

Multiplying by (-1) won't effect the convg. or divg.

So, $x_n = \frac{1}{n}$ is $\downarrow$ sequ.

and $\lim(\frac{1}{n}) = 0$

implies $\sum (-1)^n \frac{1}{n}$ convg. by A.S.T.

* Importance of Cauchy's Condition -

• Doesn't require knowing the limit in advance.

• Completeness of $\mathbb{R}$ - In $\mathbb{R}$, the converse also holds - every Cauchy sequ. converges

• Cauchy's general principle of convergence for series -

$\sum a_n$ convg. iff SOPS is Cauchy.

- Define Cauchy sequence & prove that every convergent seqn. of real numbers is a Cauchy seqn. What is the importance of the Cauchy seqn.

Soln. A sequence $x = (x_n) \in \mathbb{R}$ is stb Cauchy seqn. if $\forall \epsilon > 0 \exists$ a natural number $N(\epsilon) \neq \infty$ such that for all natural numbers $n, m \geq N(\epsilon)$, the terms x_n, x_m satisfy $|x_n - x_m| < \epsilon$

- Remember. Not every convergent sequence is a Cauchy seqn.

This is only true in $\mathbb{R}$ (or standard metric space) (or $\mathbb{R}$ is complete)

- Every Conv. seqn. in $\mathbb{R}$ is a Cauchy seqn.

Let $x = (x_n), n \in \mathbb{N}, x_n \in \mathbb{R}$ is conv. in $\mathbb{R}$

Let $\lim x_n = L$

by definition of limit $\forall \epsilon > 0 \exists K(\epsilon/2) \neq \infty$

$n \geq K(\epsilon/2)$

$$|x_n - L| < \epsilon/2$$

Define $N(\epsilon) := K(\epsilon/2)$ & $n, m \geq N(\epsilon)$

$$|x_n - x_m| = |(x_n - L) - (x_m - L)|$$

$$\leq |x_n - L| + |x_m - L| < \epsilon/2 + \epsilon/2 < \epsilon$$

$\Rightarrow$ Cauchy.

Importance - It's a limit-free definition.

• Prove that every continuous function is Riemann Integrable.

Solⁿ. ~~Correction - Every cont. function is Riemann Integrable on a close interval.~~
~~but on close interval. one close finite interval.~~

• interval needs to be close ($x = x$)

• ~~interval~~ ^{function} has to be bounded.

Defⁿ. A bounded function f on $[a, b]$ is integrable (Riemann) iff

for each $\epsilon > 0$, $\exists$ a partition P of $[a, b]$ such that -

$$U(P, f) - L(P, f) < \epsilon$$

Now, we know that

continuous function ^{finite} on close interval $[a, b]$ is uniformly continuous.

Partition $P = \{x_0, x_1, \dots, x_r, \dots, x_n = b\}$

f is uniformly cont. by definition -

$\forall \epsilon > 0 \exists \delta > 0$ for all $x, y \in [a, b]$ we have -

$$|f(x) - f(y)| < \epsilon \quad \text{whenever} \quad |x - y| < \delta$$

Now, choose partition P , with $\text{mesh}(P) = \|P\| = \max \{x_k - x_{k-1}\}$

$$< \delta$$

Now, for each sub-interval $[x_i, x_{i+1}]$, we have -

function f will attain $\max M_i$ and $\min m_i$ such that -
~~and attain at some $x_i, x_j \in [a, b]$. Then~~
 at some point say, $x_{i-1}, x_i \in [x_{i-1}, x_i]$ and this
 applies for each sub-interval.

As, we have $\text{mesh}(P) < \delta$ then -

$$M_i - m_i = f(x_i) - f(x_{i-1}) < \frac{\epsilon}{b-a}$$

so,

$$\begin{aligned} U(P, f) - L(P, f) &= \sum (M_i - m_i) \cdot \Delta x_i < \frac{\epsilon}{b-a} \cdot \sum \Delta x_i \\ &= \frac{\epsilon}{b-a} \cdot (b-a) = \epsilon \end{aligned}$$

$$\Rightarrow [U(P, f) - L(P, f) < \epsilon]$$

Now, the corrected part - why closeness & boundedness of interval matters.

- Not-closed. Take $f(x) = \frac{1}{x}$ on $(0, 1] \rightarrow$ improper int.
- Not-bounded Take $f(x) = x$ on $[0, \infty) \rightarrow \int_0^{\infty} x = \infty$

Extn - A function can be discontinuous (even everywhere, on a dense set) and still be Riemann Int.

(As long as the set of discontinuity has measure zero)

• Test the convergence of $\int_0^2 \frac{\log x}{\sqrt{2-x}} dx$

improper integral as at $x=0$, $\log x \rightarrow -\infty$

at $x=2$, $\sqrt{2-x} \rightarrow 0$

num. & deno.

split it into two parts -

$$\int_0^2 = \int_0^1 + \int_1^2$$

as both are defined at $x=1$

Now, near $x=0$, on $[0,1]$

$\frac{1}{\sqrt{2-x}}$ is bounded, implies $\int_0^1 \frac{\log x}{\sqrt{2-x}}$ entirely

depend upon the convergence of $\log x$.

$$\int_0^1 \log x dx = x \log x - x \Big|_0^1$$

as $x \rightarrow 0^+$, $x \log x \rightarrow 0$

at $x=1$, $x \log x - x = -1$

$$\Rightarrow \int_0^1 \frac{\log x}{\sqrt{2-x}} dx \text{ convg.}$$

near $x=2$ on $[1,2]$

~~$\log x$~~ $\log 2$

at $x=2$, $\frac{1}{\sqrt{2-x}}$

Substitute

$$t = 2-x$$

$$\int_1^2 \frac{\log x}{\sqrt{2-x}} dx = - \int_1^0 \frac{\log(2-t)}{\sqrt{t}} dt$$

at $t=0$, $\log(2-t) \rightarrow \log 2$

and $\int_0^1 \frac{dt}{\sqrt{t}}$ convg. by p-test as $t = \frac{1}{2} < 1$

$$\int_0^1 \frac{dx}{x^p} \text{ convg. } b < 1.$$

Both pieces converges so $\int_0^2 \frac{\log x}{\sqrt{2-x}} dx$ convg.

- Using Cauchy's general principle of convergence, examine the convergence of the seqn. $\langle f_n \rangle$, where -

$$f_n = 1 + \frac{1}{1!} + \frac{1}{2!} + \dots + \frac{1}{n!}$$

Solⁿ. fact: $e = 1 + \frac{1}{1!} + \frac{1}{2!} + \dots = \left(1 + \frac{1}{n}\right)^n$

We have to show $|f_m - f_n| < \epsilon$ $\exists m, n \geq N(\epsilon)$

WLOG, $m > n$

$$|f_m - f_n| = \frac{1}{(n+1)!} + \frac{1}{(n+2)!} + \dots + \frac{1}{m!}$$

$$\leq \frac{1}{2^n} + \frac{1}{2^{n+1}} + \dots + \frac{1}{2^{m-1}} < \frac{1}{2^{n-1}}$$

now, choose $N(\epsilon) + \frac{1}{2^{n-1}} < \epsilon$

It follows $|f_m - f_n| < \epsilon$ $\forall m, n \geq N(\epsilon) \Rightarrow$ convergence

(calculation of $N(\epsilon)$, $\frac{1}{2^{n-1}} < \epsilon \Rightarrow (n-1) \ln 2 < \ln \epsilon$

$$\Rightarrow - (n-1) \ln 2 < \ln \epsilon$$

$$\Rightarrow (n-1) > -\frac{\ln \epsilon}{\ln 2} = \frac{\ln \epsilon}{\ln \frac{1}{2}} \Rightarrow n > 1 + \frac{\ln \epsilon}{\ln \frac{1}{2}}$$

$$1 + \left\lceil \frac{\ln \epsilon}{\ln \frac{1}{2}} \right\rceil = N(\epsilon)$$

- Consider the series $\sum_{n=1}^{\infty} u_n(x)$, $0 \leq x \leq 1$, the sum of whose first n terms is given by -

$$S_n(x) = \frac{1}{2n^2} \log(1 + n^2 x^2), \quad x \in [0, 1]$$

Show that the given series can be differentiated term by term.

though $\sum_{n=1}^{\infty} u_n'(x)$ does not converge uniformly on $[0, 1]$.

Solution Underlying Theorem -

Term by term diff. theorem for series of function -

- Let $\sum u_n(x)$ be a series of diff. functions on $[a, b]$ +
 - $\sum u_n(x)$ converges at atleast one point $x_0 \in [a, b]$
 - $\sum u_n'(x)$ converges uniformly on $[a, b]$

Then -

- $\sum u_n(x)$ converges uniformly on $[a, b]$ to some funⁿ $S(x)$
- $S(x)$ is diff. on $[a, b]$
- $S'(x) = \sum u_n'(x) \quad \forall x \in [a, b]$

i.e. you're allowed to differentiate term by term.

$$\frac{d}{dx} \left(\sum u_n(x) \right) = \sum \frac{d}{dx} u_n(x)$$

$S_n(x) = \frac{1}{2n^2} \log(1 + n^2 x^2)$, $x \in [0, 1]$, where $S_n(x)$ is the sum of the first n terms of $\sum u_n(x)$

$$S_n(x) = \frac{1}{2n^2} \log(1+n^4 x^2) \quad [0,1]$$

$$x=0 \Rightarrow S_n(0) = 0$$

$$x \in (0,1) \text{ as } n \rightarrow \infty$$

$$\text{See, } \log(1+n^4 x^2) \sim \log(n^4 x^2) = 4 \log n + 2 \log x$$

$$\Rightarrow S_n(x) \sim \frac{4 \log n}{2n^2} \xrightarrow{n \rightarrow \infty} 0$$

$$\text{Hence } \lim_{n \rightarrow \infty} S_n(x) = S(x) = 0 \quad \forall x \in [0,1]$$

$$\Rightarrow S'(x) = 0$$

Now, differentiate term by term - (computing $S'_n(x)$)

$$S'_n(x) = \frac{1}{2n^2} \cdot \frac{2n^4 x}{1+n^4 x^2} = \frac{n^2 x}{1+n^4 x^2}$$

now, for pointwise convergence of $S'_n(x)$ -

$$\text{At } x=0, S'_n(x) = 0$$

$$x \in (0,1) \text{ as } n \rightarrow \infty$$

$$S'_n(x) = \frac{n^2 x}{1+n^4 x^2} \sim \frac{n^2 x}{n^4 x^2} = \frac{1}{n^2 x} \rightarrow 0$$

$$\text{So, } S'_n(x) \rightarrow 0 = S'(x) \text{ pointwise on } [0,1]$$

This confirms the term-by-term differentiation is valid.

$S_n'(x)$ does not converge uniformly.

To show $\sup_{x \in [0,1]} |S_n'(x) - 0| \rightarrow 0$

Maximize, $S_n'(x) = \frac{n^2 x}{1+n^4 x^2}$ on $[0,1]$

Let $f(x) = \frac{n^2 x}{1+n^4 x^2}$

$f'(x) = \frac{n^2(1-n^4 x^2)}{(1+n^4 x^2)^2}$, Setting $f'(x) = 0$

$x = \frac{1}{n^2} \in [0,1] \forall n$

Now,

$f\left(\frac{1}{n^2}\right) = \frac{n^2 \cdot \frac{1}{n^2}}{1+n^4 \cdot \frac{1}{n^4}} = \frac{1}{1+1} = \frac{1}{2}$

So, $\sup_{x \in [0,1]} |S_n'(x)| = \frac{1}{2} \nrightarrow 0$

this imply - $\sup_{x \in [0,1]} |S_n'(x) - S'(x)| = \frac{1}{2} \nrightarrow 0$

Therefore, $\sum u_n(x)$ does not converge uniformly on $[0,1]$,

even though it converges pointwise to the correct derivative in $S'(x) = 0$.

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Key: Theorem: condition (uniform condition of derivatives) is not sufficient not necessary.

Term by term diff. works even when uniform convergence fails

$$[1, 0] = I$$

Find the upper and lower Riemann integrals for the function f defined on $[0, 1]$ as follows -

$$f(x) = \begin{cases} \sqrt{1-x^2} & x \in \mathbb{Q} \\ 1-x & x \in \mathbb{Q}^c \end{cases}$$

show that f is not Riemann integrable on $[0, 1]$

Solⁿ define $f(x) = \begin{cases} g(x) & x \in \mathbb{Q} \cap [0, 1] \\ h(x) & x \in \mathbb{Q}^c \cap [0, 1] \end{cases}$

Now, $(g^2 - h^2)(x) = g^2(x) - h^2(x) = (1-x)^2 - (1-x^2)$
 $= (1-x)(1+x) - (1-x)(1+x) = (1-x)[1+x - (1-x)]$
 $= 2x(1-x) \geq 0$

equality at $x=0$ & $x=1$, $g(x) \geq h(x) \geq 0$ on $[0, 1]$

It gives $g(x) \geq h(x) \forall x \in [0, 1]$ equality only at $x=0, x=1$

Now, Supremum & Infimum of f on any sub-interval Δx . ~~is same~~

As, rational & irrational are ~~same~~ both dense in this sub-interval.

And, know that $g(x)$ and $h(x)$ are the restriction of $f(x)$ on rationals & irrationals.

$$I = [0, 1]$$

$$\Rightarrow \sup_{x \in \mathbb{Q}} g(x) = \sup_{x \in I} g(x) \quad \& \quad \inf_{x \in \mathbb{Q}^c} h(x) = \inf_{x \in I} h(x)$$

For $x \in \mathbb{Q}$, $f(x) = g(x) \leq \sup_{x \in I} g(x)$

$x \in \mathbb{Q}^c$, $f(x) = h(x) \leq g(x) \leq \sup_{x \in I} g(x)$

Hence $\sup_{x \in I} f(x) \leq \sup_{x \in I} g(x)$

Conversely $\sup_{x \in I} f(x) \geq \sup_{x \in \mathbb{Q}} f(x) = \sup_{x \in \mathbb{Q}} g(x) = \sup_{x \in I} g(x)$

imply $\left[\sup_{x \in I} f(x) = \sup_{x \in I} g(x) \right]$

• Symmetrically,

$x \in \mathbb{Q}^c$

$\left[\inf_{x \in I} f(x) = \inf_{x \in I} h(x) \right]$

Now,

$$U(f, P) = \sum_i \left(\sup_{\Delta x_i} f \right) \Delta x_i = \sum_i \left(\sup_{\Delta x_i} g \right) \Delta x_i = U(g, P) \quad U(f, P) \neq U(g, P)$$

$$L(f, P) = \sum_i \left(\inf_{\Delta x_i} f \right) \Delta x_i = \sum_i \left(\inf_{\Delta x_i} g \right) \Delta x_i = L(g, P)$$

$$U(f) = \int_0^1 f(x) dx = \inf_P U(f, P) = \inf_P U(g, P) = \int_0^1 g(x) dx$$

$$= \int_0^1 \sqrt{1-x^2} dx = \frac{\pi}{4}$$

$$U(L) = \int_0^1 f(x) dx = \sup_P L(f, P) = \sup_P L(g, P) = \int_0^1 (1-x) dx = \frac{1}{2}$$

This shows that f is not Riemann Integrable as $U(f) \neq L(f)$.

• Test the convergence of the series -

$$\sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdot 5 \dots (2n-1)}{2 \cdot 4 \cdot 6 \dots (2n)} \cdot \frac{x^{2n+1}}{2n+1}, \quad x > 0$$

lets Define this by $\sum a_n$, where

$$a_n = \frac{1 \cdot 3 \cdot 5 \dots (2n-1)}{2 \cdot 4 \cdot 6 \dots (2n)} \cdot \frac{x^{2n+1}}{2n+1}$$

and $a_n = c_n \cdot \frac{x^{2n+1}}{2n+1}$, $c_n = \frac{1 \cdot 3 \dots (2n-1)}{2 \cdot 4 \dots (2n)}$

$$x > 0 \Rightarrow a_n > 0 \Rightarrow \left| \frac{a_{n+1}}{a_n} \right| = \frac{a_{n+1}}{a_n}$$

• Ratio Test - $\frac{a_{n+1}}{a_n} = \frac{c_{n+1}}{c_n} \cdot x^2 \cdot \frac{2n+1}{2n+3}$

now, $\frac{c_{n+1}}{c_n} = \frac{2n+1}{2n+2}$

so, $\frac{a_{n+1}}{a_n} = \frac{2n+1}{2n+2} \cdot x^2 \cdot \frac{2n+1}{2n+3} = x^2 \cdot \frac{(2n+1)^2}{(2n+2)(2n+3)}$

$$\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = x^2$$

• converges for $x^2 < 1 \Rightarrow 0 < x < 1$ ($x > 0$)

• diverges for $x^2 > 1 \Rightarrow x > 1$

• inconclusive exactly at $x = 1$ -

Now, for $x=1$ (Use ~~Rat~~ Raabe's Test)

At $x=1$, $a_n = \frac{C_n}{2n+1}$

• Raabe's Test - For a series of +ve terms $\sum a_n$, if $\lim_{n \rightarrow \infty} n \left(\frac{a_n}{a_{n+1}} - 1 \right) = L$, Exists... then the series

• Converges if $L > 1$

• Diverges if $L < 1$

• Inconclusive at $L = 1$

Now, $\frac{a_n}{a_{n+1}} = \frac{(2n+2)(2n+3)}{(2n+1)^2}$

$$\frac{a_n}{a_{n+1}} - 1 = \frac{(2n+2)(2n+3) - (2n+1)^2}{(2n+1)^2} = \frac{6n+5}{(2n+1)^2}$$

$$\Rightarrow n \left(\frac{a_n}{a_{n+1}} - 1 \right) = \frac{n(6n+5)}{(2n+1)^2} = \frac{6n^2+5n}{4n^2+4n+1}$$

$$\Rightarrow \lim_{n \rightarrow \infty} n \left(\frac{a_n}{a_{n+1}} - 1 \right) = \lim_{n \rightarrow \infty} \frac{6n^2+5n}{4n^2+4n+1} = \frac{3}{2} > 1$$

$\Rightarrow$ series converges at $x=1$

Conclusion - converges for $0 < x \leq 1$ and diverges for $x > 1$

3. • Test for convergence or divergence of the series -

$$x + \frac{2^2 x^2}{2!} + \frac{3^3 x^3}{3!} + \dots \quad (x > 0)$$

It can be written as -

$$\sum_{n=1}^{\infty} \frac{n^n x^n}{n!}, \quad a_n = \frac{n^n x^n}{n!}$$

$$a_n > 0 \text{ as } x > 0$$

Ratio Test - $\left| \frac{a_{n+1}}{a_n} \right| = \frac{a_{n+1}}{a_n} = \frac{(n+1)^{n+1} \cdot x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n n^n} = \frac{(n+1)^n}{n^n} \cdot x$

$$\Rightarrow \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n} \right)^n \cdot x = ex$$

Converges for $ex < 1$ i.e. $x < \frac{1}{e}$

Diverges for $ex > 1$ i.e. $x > \frac{1}{e}$

Inconclusive for $x = \frac{1}{e}$

Now, for $\left[x = \frac{1}{e} \right]$ $a_n = \frac{n^n \cdot e^{-n}}{n!}$

using $n! \sim n^n \cdot e^{-n} \cdot \sqrt{2\pi n}$

$$a_n = \frac{n^n (1/e)^n}{n!} \sim \frac{n^n \cdot e^{-n}}{n^n \cdot e^{-n} \cdot \sqrt{2\pi n}} = \frac{1}{\sqrt{2\pi n}} \sim \frac{1}{\sqrt{n}}$$

Now, compare the p-series $\sum \frac{1}{\sqrt{n}}$, $p = \frac{1}{2} \leq 1 \Rightarrow$ diverges

By the limit comparison Test

$$(0 < x) \quad \frac{a_n}{b_n} \rightarrow L \quad L \neq 0 < \infty \Rightarrow \text{converges}$$

$$a_n = \frac{1}{\sqrt{2n}} \quad b_n = \frac{1}{\sqrt{n}}$$

$$\frac{a_n}{b_n} = \frac{\frac{1}{\sqrt{2n}}}{\frac{1}{\sqrt{n}}} = \frac{1}{\sqrt{2}} \neq 0 \Rightarrow \sum a_n \text{ diverges as } \sum b_n$$

diverges as p-series diverges for $p \leq 1$

$$x = \frac{1}{n} \Rightarrow \left| \frac{1}{n} \right| \rightarrow 0 \Rightarrow \left| \frac{1}{n} \right| \rightarrow 0$$

Conclusion-

converges for $0 < x < \frac{1}{e}$

$$\frac{1}{e} > x$$

diverges for $x \geq \frac{1}{e}$

$$\frac{1}{e} < x$$

$$\frac{1}{e} = x$$

$$\frac{1}{e} = x$$

$$\left[\frac{1}{e} = x \right] \text{ not work}$$

$$\frac{1}{e} = x$$

$$\frac{1}{e} = x$$

$$\frac{1}{e} = x$$

$$\frac{1}{e} = x$$

$$\frac{1}{e} = x$$

now, compare the p-series $\sum \frac{1}{n^p}$ $p \leq 1 \Rightarrow$ diverges

- Prove the following oscillation of a real-valued bounded function f defined on $[a, b]$ is the supremum of the set $\{ |f(x_1) - f(x_2)| : x_1, x_2 \in [a, b] \}$

Solⁿ:

statement - let $f: [a, b] \rightarrow \mathbb{R}$ be a bounded real-valued funⁿ.

The oscillation of f on $[a, b]$, defined as -

$$w(f) = M - m, \quad M = \sup_{[a, b]} f(x), \quad m = \inf_{[a, b]} f(x)$$

what we have to prove -

$$w(f) = \sup \{ |f(x_1) - f(x_2)| : x_1, x_2 \in [a, b] \}$$

let $S = \{ |f(x_1) - f(x_2)| : x_1, x_2 \in [a, b] \}$. Since f is bounded,

M & m exist and finite by the completeness property of $\mathbb{R}$.

Step-1: $M - m$ is an upper bound of S .

pick any $x_1, x_2 \in [a, b]$. $m \leq \underline{f(x_1) \wedge f(x_2)} \leq M$

$$\text{therefor } |f(x_1) - f(x_2)| \leq M - m$$

implies $M - m$ is an upper bound of S , and since $S \neq \emptyset$ and bound above, $\sup S$ exists i.e.

$$\sup S \leq M - m \quad \text{--- ①}$$

Step-2: $M-m$ is the least upper bound

Given any $\epsilon > 0$, $\exists x_1$ and $x_2 \in [a, b]$ such that -

$$f(x_1) > M - \frac{\epsilon}{2}$$

$$\text{and } f(x_2) < m + \frac{\epsilon}{2}$$

Then

$$f(x_1) - f(x_2) > (M - \frac{\epsilon}{2}) - (m + \frac{\epsilon}{2}) = M - m - \epsilon$$

$\Rightarrow |f(x_1) - f(x_2)| > M - m - \epsilon$, which is an element of S . Hence -

$$\sup S \geq M - m \quad \text{--- (2)}$$

Conclusion - combining (1) and (2), we have -

$$\sup S = M - m = \sup_{[a, b]} f - \inf_{[a, b]} f = w(f)$$

$$\Rightarrow w(f) = \sup_{x_1, x_2 \in [a, b]} |f(x_1) - f(x_2)|$$

$$m - M \geq |f(x_1) - f(x_2)|$$

①

$$m - M \geq 2 \epsilon$$

- Test the convergence of $\int_0^{\infty} \frac{\cos x}{1+x^2} dx$

Consider

$$I(b) = \int_0^b \frac{\cos x}{1+x^2} dx$$

$$I(b) = \int_0^b \frac{\cos x}{1+x^2} dx$$

$$\int_0^{\infty} \frac{\cos x}{1+x^2} dx = \lim_{b \rightarrow \infty} \int_0^b \frac{\cos x}{1+x^2} dx$$

This is type-1, improper integral as (infinite interval).

The integrand $\frac{\cos x}{1+x^2}$ is continuous on $[0, \infty)$ (since $1+x^2 \neq 0$)

so there's no issue at any interior point.

Only point we need to check is at upper limit.

- We'll use comparison test and show that this converges absolutely—

$$\forall x \geq 0, |\cos x| \leq 1 \Rightarrow \left| \frac{\cos x}{1+x^2} \right| \leq \frac{1}{1+x^2}$$

$$\text{Now, } \int_0^{\infty} \frac{\cos x}{1+x^2} dx = \lim_{b \rightarrow \infty} \int_0^b \frac{\cos x}{1+x^2} dx \leq \lim_{b \rightarrow \infty} \int_0^b \frac{dx}{1+x^2}$$

$$\begin{aligned} \text{and } \lim_{b \rightarrow \infty} \int_0^b \frac{dx}{1+x^2} &= \lim_{b \rightarrow \infty} (\arctan b - \arctan 0) \\ &= \frac{\pi}{2} - 0 = \frac{\pi}{2} < \infty \end{aligned}$$

$$\Rightarrow \int_0^{\infty} \left| \frac{\cos x}{1+x^2} \right| dx \text{ converges}$$

this further implies —

$$\int_0^{\infty} \frac{\cos x}{1+x^2} dx \text{ converges absolutely.}$$

Extra-

Using Residue and Contour Integration,

$$\int_0^{\infty} \frac{\cos x}{1+x^2} dx = \frac{\pi}{2e}$$

only point we need to check is at upper limit.
to check? no limit at upper limit.
the integrand is continuous on $[0, \infty)$ (since $1+x^2 \neq 0$).
This is typical. (infinite interval).

we'll use comparison test and show that the integral converges.

$$x \geq 0, \quad \left| \frac{\cos x}{1+x^2} \right| \leq \frac{1}{1+x^2}$$

$$\lim_{p \rightarrow \infty} \int_0^p \frac{\cos x}{1+x^2} dx = \lim_{p \rightarrow \infty} \int_0^p \frac{1}{1+x^2} dx$$

$$\lim_{p \rightarrow \infty} \left(\frac{1}{1+x^2} \right) = \lim_{p \rightarrow \infty} \left(\frac{1}{1+p^2} \right) = 0$$

$$\frac{\pi}{2} - 0 = \frac{\pi}{2} > 0$$

let $f(x) = x^2$ on $[0, k]$, $k > 0$. show that f is Riemann integrable on the above interval $[0, k]$

and $\int_0^k f(x) dx = \frac{k^3}{3}$

solⁿ. Take the uniform partition P_n of $[0, k]$ in n equal parts

$$x_i = \frac{ik}{n}, i=0, 1, 2, \dots, n, \Rightarrow \Delta x_i = \frac{k}{n}$$

where $P = \{x_0, x_1, \dots, x_n\}$

Since $f(x) = x^2$ is increasing on $[0, k]$, on each sub interval $[x_{i-1}, x_i]$

$$M_i = f(x_i) = \left(\frac{ik}{n}\right)^2, \quad m_i = f(x_{i-1}) = \left(\frac{(i-1)k}{n}\right)^2$$

upper sum -

$$U(P_n, f) = \sum_{i=0}^n m_i \Delta x_i = \sum_{i=0}^n \left(\frac{ik}{n}\right)^2 \cdot \frac{k}{n} = \frac{k^3}{n^3} \sum_{i=0}^n i^2$$

$$\Rightarrow U(P_n, f) = \frac{k^3}{n^3} \cdot \frac{n(n+1)(2n+1)}{6}$$

similarly -

$$L(P_n, f) = \sum_{i=0}^n m_i \Delta x_i = \frac{k^3}{n^3} \sum_{i=0}^n \frac{(i-1)(i)(2i-1)}{6}$$

$$\Rightarrow L(P_n, f) = \sum_{i=0}^n \left(\frac{(i-1)k}{n}\right)^2 \cdot \frac{k}{n} = \frac{k^3}{n^3} \sum_{i=0}^n (i-1)^2$$

$$\Rightarrow L(P_n, f) = \frac{k^3}{n^3} \cdot \frac{(n-1)(n)(2n-1)}{6}$$

Now by the definition, f on $[a, b]$, f is integrable

if $\forall \epsilon > 0, \exists P \rightarrow$

$$U(P, f) - L(P, f) < \epsilon$$

Consider

$$U(p_n, f) - L(p_n, f) = \frac{K^3}{6n^2} [(n+1)(2n+1) - (0-1)(2n-1)]$$

$$= \frac{K^3}{6n^2} [6n] = \frac{K^3}{n}$$

Now, given any $\epsilon > 0$, choose n large enough that $\frac{K^3}{n} < \epsilon$

then $U(p_n, f) - L(p_n, f) < \epsilon$

f is Riemann Integrable.

Calculation of n .

$$\frac{K^3}{n} < \epsilon$$

$$\Rightarrow n > \frac{K^3}{\epsilon}$$

choose

$$m = \left[\frac{K^3}{\epsilon} \right] + 1$$

and

$\forall n \geq m, \frac{K^3}{n} < \epsilon$ holds

$[.]$ is g.f

so, choose

$$P = p_m$$

for $m = \left[\frac{K^3}{\epsilon} \right] + 1$

Then, By definition it holds—

• Values of Integrable - We have proved f is integrable then

$$L(f) = \sup L(p, f) = \inf U(p, f) = U(f)$$

$$U(f) = \lim_{n \rightarrow \infty} U(p_n, f) = \frac{K^3(2n^2 + 3n + 1)}{6n^2} \rightarrow \frac{K^3}{3} \text{ as } n \rightarrow \infty$$

$$\text{so, } \int_0^K x^2 dx = \frac{K^3}{3}$$

- Test the uniform convergence of the series —

$$x^4 + \frac{x^4}{1+x^4} + \frac{x^4}{(1+x^4)^2} + \dots \quad \text{on } [0,1]$$

Solution —

$$\sum_{n=1}^{\infty} f_n(x), \quad \text{where } f_n(x) =$$

~~series~~

$$\sum_{n=0}^{\infty} \frac{x^4}{(1+x^4)^n} \quad \text{on } [0,1], \quad \text{series of function}$$

Concepts used —

- each term $f_n(x) = \frac{x^4}{(1+x^4)^n}$ is continuous on $[0,1]$ as denominator never vanishes, so every partial sum $S_N = \sum_{n=0}^N f_n(x)$ is continuous. If the series converged uniformly, the theorem "a uniform limit of continuous functions is continuous" would force the sum $S(x)$ to be continuous too.

So, we have to find the pointwise limit sum and check whether it's continuous. If it isn't uniform convergence fails.

- Compute $M_N(x) = S(x) - S_N(x)$, and check whether $\sup_x |M_N(x)| \rightarrow 0$, $x \in [0,1]$.

If this does not tend to 0 for all $x \in [0,1]$, then it is not uniformly convergent.

Computing pointwise sum $s(x)$

$$\text{At } x=0, f_n(0) = \frac{0^n}{(1+0)^n} = 0 \Rightarrow s(0) = 0$$

$$\text{for } x \in (0,1] \quad \delta = \frac{1}{1+x^4} \quad \text{as } x^4 > 0 \Rightarrow 0 < \delta < 1$$

$$\Rightarrow s(x) = \sum_{n=0}^{\infty} x^4 \cdot \delta^n = x^4 \cdot \frac{1}{1-\delta} = x^4 \cdot \frac{1}{1 - \frac{1}{1+x^4}}$$

$$\Rightarrow s(x) = 1+x^4$$

$$\text{point wise sum, } s(x) = \begin{cases} 1+x^4, & x \in (0,1] \\ 0, & x=0 \end{cases}$$

We can easily see that $s(x)$ is not continuous at $x=0$

$$\lim_{x \rightarrow 0^+} s(x) = \lim_{x \rightarrow 0^+} (1+x^4) = 1 \neq 0 = s(0)$$

$\Rightarrow$ discontinuity at $x=0$.

even though every f_n and $s_n(x)$ is continuous on $[0,1]$

so, By the continuity theorem for uniform limits, this alone proves the series does not converge uniformly on $[0,1]$.

• We can also use this $M_N(x)$ method -

$$M_N(x) = S(x) - S_N(x)$$

so, if $\sup |M_N(x)| \rightarrow 0$ as $N \rightarrow \infty \quad \forall x \in [0,1]$

then convergence is uniform.

$$\text{At, } x=0 \quad M_N(x) = M_N(0) = S(0) - S_N(0) = 0$$

for $x \in (0,1]$:

$$M_N(x) = S(x) - S_N(x) = x^4 \sum_{n=N+1}^{\infty} x^n$$

$$\Rightarrow M_N(x) = x^4 \cdot \frac{x^{N+1}}{1-x} = \frac{1}{(1+x)^N}$$

Now, taking the supremum of $|M_N(x)|$

$$\sup \{ |M_N(x)| : x \in [0,1] \} = \sup_{(0,1]} \frac{1}{(1+x)^N}$$

as $x \rightarrow 0^+$, $\frac{1}{(1+x)^N} \rightarrow 1$, and this value is never

attained for $x > 0$ but approached arbitrarily closely. So —

$$\sup_{[0,1]} |M_N(x)| = 1 \quad \forall N \in \mathbb{N}$$

By using above statement, supremum does not tend to 0 as $N \rightarrow \infty$ ($=1$). Hence it's not uniformly convergent on $[0,1]$

But it is uniformly const. for $x \in [\delta, 1]$, $\delta > 0$ (why?? check)

- If a function f is monotonic on the interval $[a, b]$, then prove that f is Riemann integrable on $[a, b]$.

Sol. without loss of generality (wlog) assume f is non-decreasing
 since f is non-decreasing on $[a, b]$

$$f(a) \leq f(x) \leq f(b) \quad \forall x \in [a, b]$$

so, f is bdd. $\Rightarrow (f)$ is bounded

- f is integrable on $[a, b]$ iff for every $\epsilon > 0$, $\exists$ a partition P of $[a, b]$ s.t.
 $U(f, P) - L(f, P) < \epsilon$

Now, the construction of partition -

let $\epsilon > 0$, be given. choose $n \in \mathbb{N}$ large enough that -

$$\frac{(b-a)(f(b)-f(a))}{n} < \epsilon$$

$$\left[\begin{array}{l} \frac{(b-a)(f(b)-f(a))}{\epsilon} < n \\ \text{choose } n = \left\lceil \frac{(b-a)(f(b)-f(a))}{\epsilon} \right\rceil + 1 \end{array} \right] \text{ then true for all } \epsilon.$$

and by Archimedean prop. we can find such n .

let $P = \{x_0, x_1, \dots, x_n\}$ be the uniform partition of $[a, b]$ into n equal

subintervals: $x_i = a + i \cdot \frac{b-a}{n}$ $0 \leq i \leq n$, $\Delta x_i = \frac{b-a}{n}$

Computing m_i & M_i on each subinterval. Since f is non-decreasing, on each subinterval $[x_{i-1}, x_i]$, the infimum & supremum are simply attained at the endpoints:

$$m_i = \inf_{[x_{i-1}, x_i]} f = f(x_{i-1}), \quad M_i = \sup_{[x_{i-1}, x_i]} f = f(x_i)$$

Consider,

$$\begin{aligned} U(f, P) - L(f, P) &= \sum_{i=1}^n (M_i - m_i) \Delta x_i = \sum_{i=1}^n [f(x_i) - f(x_{i-1})] \frac{b-a}{n} \\ &= \frac{b-a}{n} \cdot \sum_{i=1}^n [f(x_i) - f(x_{i-1})] \end{aligned}$$

Now the telescoping sum-

$$\begin{aligned} \sum_{i=1}^n [f(x_i) - f(x_{i-1})] &= [f(x_1) - f(x_0)] + [f(x_2) - f(x_1)] + \dots \\ &\quad \dots + [f(x_n) - f(x_{n-1})] \\ &= f(x_n) - f(x_0) = f(b) - f(a) \end{aligned}$$

Therefore:

$$U(f, P) - L(f, P) = \frac{b-a}{n} [f(b) - f(a)] = \frac{b-a}{n} [f(b) - f(a)] < \epsilon$$

by the choice of n .

Conclusion-

$\forall \epsilon > 0$, we have formed a partition P (the uniform partition) such that $U(f, P) - L(f, P) < \epsilon$

Hence by Riemann Criterion, f is Riemann Integrable on $[a, b]$

• Similarly it follows for non-increasing funⁿ.

Remarks-

A monotonic function can have infinitely many discontinuities and these discontinuities are at most countable.

• Discontinuities are called jump discontinuities.

Question- Prove that the sequence $\langle a_n \rangle$ satisfying the condition-

$$|a_{n+1} - a_n| \leq \alpha |a_n - a_{n-1}|, \quad 0 < \alpha < 1 \quad \forall n \geq 2$$

is a Cauchy sequence -

Solⁿ.

Define $d_n = |a_{n+1} - a_n|$, $n = 1, 2, \dots$

then the condition becomes -

$$d_n \leq \alpha \cdot d_{n-1} \quad \forall n \geq 2$$

Claim-

$$d_n \leq \alpha^{n-1} d_1 \quad \forall n \geq 1$$

Proof by Mathematical Induction -

for $n=1$,

$$d_1 \leq \alpha^{1-1} d_1 = \alpha^0 d_1 = d_1$$

equality holds trivially -

for $n=k$, suppose the claim holds

$$\text{i.e. } d_k \leq \alpha^{k-1} d_1$$

for $n=k+1$, we must show the claim holds for $n=k+1$

$$\text{i.e. } d_{k+1} \leq \alpha^k d_1$$

$$d_{k+1} \leq \alpha d_k$$

now by our inductive hypothesis $d_k \leq \alpha^{k-1} d_1$

$$\text{Now, } d_{k+1} \leq \alpha d_k \leq \alpha (\alpha^{k-1} d_1) = \alpha^k d_1$$

$$\Rightarrow d_{k+1} \leq \alpha^k d_1$$

So, by Principle of Mathematical Induction, the claim -

$$d_n \leq \alpha^{n-1} d_1 \text{ holds for all } n \geq 1.$$

Now, -

$|a_m - a_n|$ for arbitrary $m > n$.

$$a_m - a_n = (a_{n+1} - a_n) + (a_{n+2} - a_{n+1}) + \dots + (a_m - a_{m-1})$$

$$\Rightarrow |a_m - a_n| = \left| \sum_{k=n}^{m-1} (a_{k+1} - a_k) \right| \leq \sum_{k=n}^{m-1} |a_{k+1} - a_k| = \sum_{k=n}^{m-1} d_k$$

Substituting the geometric bound -

$$d_k \leq \alpha^{k-1} d_1$$

$$\text{i.e. } |a_{k+1} - a_k| \leq \alpha^{k-1} |a_2 - a_1|$$

we have -

$$|a_m - a_n| \leq \sum_{k=n}^{m-1} \alpha^{k-1} d_1 = |a_2 - a_1| \cdot \sum_{k=n}^{m-1} \alpha^{k-1}$$

$$\Rightarrow |a_m - a_n| \leq |a_2 - a_1| \cdot \alpha^{n-1} \cdot \left(\frac{1 - \alpha^{m-n}}{1 - \alpha} \right)$$

$$\Rightarrow |a_m - a_n| \leq \alpha^{n-1} \cdot \left(\frac{1}{1 - \alpha} \right) \cdot (|a_2 - a_1|)$$

and we can make it less than ϵ by selecting n , where

n is -

$$\frac{\alpha^{n-1}}{1-\alpha} \cdot |q_2 - q_1| < \varepsilon \Rightarrow \alpha^{n-1} < \frac{\varepsilon \cdot (1-\alpha)}{|q_2 - q_1|}$$

$$\log \alpha^{n-1} < \log \left(\frac{\varepsilon \cdot (1-\alpha)}{|q_2 - q_1|} \right)$$

$$\Rightarrow n-1 > \frac{\log \left(\frac{\varepsilon \cdot (1-\alpha)}{|q_2 - q_1|} \right)}{\log \alpha} \quad \begin{matrix} \log \alpha < 0 \\ \text{as } 0 < \alpha < 1 \end{matrix}$$

$$\Rightarrow n > \frac{\log \left(\frac{\varepsilon \cdot (1-\alpha)}{|q_2 - q_1|} \right)}{\log \alpha}$$

choose $n_0 = \left\lceil \frac{\log \left(\frac{\varepsilon \cdot (1-\alpha)}{|q_2 - q_1|} \right)}{\log \alpha} \right\rceil + 1$

So for all $n > n_0$ it is true

So, ~~for~~, for any given $\varepsilon > 0$, and any $m > n \geq n_0$.

we have

$$|q_m - q_n| < \varepsilon \quad \forall m, n \geq n_0$$

• show that the function -

$$f(x, y) = \begin{cases} \frac{x^2 - y^2}{x - y} & (x, y) \neq (1, -1) \\ 0 & (x, y) = (1, -1) \end{cases}$$

is continuous and differentiable at $(1, -1)$.

Soln.

We will use the standard theorem, "if the partial derivatives exist in a nbd. of a point and are continuous there, then f is diff. at that point."

(1) First observe that $(1, -1)$ and $(1, 1)$ are at a distance of $\sqrt{2}$.

So, we can choose a small nbd. of $(1, -1)$

$$N_\epsilon(1, -1) = \left\{ (x, y) \in \mathbb{R} \times \mathbb{R} \mid \sqrt{(x-1)^2 + (y+1)^2} < \frac{\epsilon}{\sqrt{2}} \right\}$$

$\epsilon < \sqrt{2}$.

Now,

Inside this nbd. we can

$$f(x, y) = \frac{(x-y)(x+y)}{(x-y)} = x+y \quad \text{as } (x, y) \neq (1, -1)$$

i.e. $f(x, y) = x+y$ throughout an entire nbd of $(1, -1)$ including $(1, -1)$ itself.

It means that, we can easily check the continuity of f at $(1, -1)$.

We must show - $\lim_{(x, y) \rightarrow (1, -1)} f(x, y) = f(1, -1) = 0$

$$\lim_{(x, y) \rightarrow (1, -1)} f(x, y) = x+y \Big|_{(1, -1)} = 1 + (-1) = 0$$

so this shows the continuity of $f(x, y)$ at point $(1, -1)$

for differentiability,

we must show f_x and f_y exist and are continuous in the nbd. of $(1, -1)$.

Partial derivative f_x, f_y using first principles-

$$f_x(1, -1) = \lim_{h \rightarrow 0} \frac{f(1+h, -1) - f(1, -1)}{h} = \lim_{h \rightarrow 0} \frac{h-0}{h} = 1$$

$$f_y(1, -1) = \lim_{k \rightarrow 0} \frac{f(1, -1+k) - f(1, -1)}{k} = \lim_{k \rightarrow 0} \frac{k-0}{k} = 1$$

for small h and k , $(1+h, -1)$ and $(1, -1+k)$ lies in the nbd.

and is not equal to $(1, -1)$ as $h \neq 0, k \neq 0$.

Both partial derivative exists - a necessary condition for differentiability. But not sufficient.

Sufficient condition is Existence of partial derivatives and are continuous at point.

$$\text{We have } f_x(1, -1) = 1, \quad f_y(1, -1) = 1 \quad \forall (x, y) \in N_\epsilon(1, -1)$$

(constant in the nbd.)

and constant functions are continuous

Hence, partial derivatives are continuous

Therefore, by necessary & sufficient condition $f(x, y)$ is differentiable at $(1, -1)$.

Discuss the uniform convergence of $f_n(x) = \frac{nx}{1+n^2x^2}$, $\forall x \in \mathbb{R} (-\infty, \infty)$

Solⁿ $f_n(x) = \frac{nx}{1+n^2x^2}$

first find the pointwise limit -

At $x=0$, $f_n(0) = \frac{n \cdot 0}{1+n^2 \cdot 0^2} = \frac{0}{1} = 0 \quad \forall n \in \mathbb{N}$

for $x \neq 0$, $f_n(x) = \frac{nx}{1+n^2x^2} = \frac{x/n}{\frac{1}{n^2} + x^2}$

observe, As $n \rightarrow \infty$, $\frac{x}{n} \rightarrow 0$ while the denominator $\frac{1}{n^2} + x^2 \rightarrow x^2$

i.e. $f_n(x) \rightarrow \frac{0}{x^2} = 0$

Conclusion- For every $x \in \mathbb{R}$, $f_n(x) \rightarrow 0$. So the pointwise limit function $f(x) \equiv 0$.

Now, we have to check whether the convergence is uniform -

By definition- $f_n \rightarrow f$ uniformly ^{on $\mathbb{R}$} means that -

$$\sup_{x \in \mathbb{R}} |f_n(x) - f(x)| \rightarrow 0 \text{ as } n \rightarrow \infty$$

So, we need to compute $\sup_{x \in \mathbb{R}} |f_n(x)|$ exactly for each fixed n .

now finding the maximum of $|f_n(x)|$ - $|f_n|$ is symmetric about y-axis.

$$f_0(x) = \frac{nx}{1+n^2x^2}$$

so, we'll find x for $x > 0$.

$$f'_0(x) = \frac{n(1+n^2x^2) - nx(2n^2x)}{(1+n^2x^2)^2} = \frac{n(1-n^2x^2)}{(1+n^2x^2)^2}$$

setting $f'_0(x) = 0$

$$n(1-n^2x^2) = 0 \Rightarrow 1 - n^2x^2 = 0 \Rightarrow x^2 = \frac{1}{n^2} \Rightarrow x = \pm \frac{1}{n}$$

checking maximum for $0 < x < \frac{1}{n}$, $1 - n^2x^2 > 0$, so $f'_0(x) > 0$

and for $x > \frac{1}{n}$, $1 - n^2x^2 < 0$, so $f'_0(x) < 0$

implies maximum at $x = \frac{1}{n}$.

so, Maximum Value of $f_n(x)$ is $f_n\left(\frac{1}{n}\right)$

$$f_n\left(\frac{1}{n}\right) = \frac{n \cdot \frac{1}{n}}{1 + n^2 \cdot \frac{1}{n^2}} = \frac{1}{2}$$

Therefore: $\sup_{x \in \mathbb{R}} |f_n(x)| = \frac{1}{2} \quad \forall n \geq 1$

Now, the uniform convergence condition -

$$\sup_{x \in \mathbb{R}} |f_n(x) - f(x)| = \frac{1}{2} \quad \forall n$$

which is not tending towards zero.

Hence, $f_n(x)$ is not uniformly convergent.

• Discuss the convergence of $\int_1^2 \frac{\Gamma x}{\ln x} dx$

Solⁿ first observe that $f(x) = \frac{\Gamma x}{\ln x}$ is continuous on $(1, 2]$, but as $x \rightarrow 1^+$, we have $\ln x \rightarrow 0$, so the integrand becomes unbounded near $x=1$.

So, we need to test the improper integral -

$$\int_1^2 \frac{\Gamma x}{\ln x} dx$$

for convergence at the lower limit $x=1$.

As $x \rightarrow 1^+$

$\Gamma x \rightarrow 1$

$\ln x \sim x-1$

$$\left(\lim_{x \rightarrow 1} \frac{\ln x}{x-1} = 1 \right)$$

define

$$g(x) = \frac{1}{x-1}$$

Now, use limit comparison Test -

$$\lim_{x \rightarrow 1^+} \frac{f(x)}{g(x)} = \lim_{x \rightarrow 1^+} \frac{\frac{\Gamma x}{\ln x}}{\frac{1}{x-1}} = 1 \neq 0, < \infty$$

Therefore, by the limit comparison Test -

$$\int_1^2 \frac{\Gamma x}{\ln x} dx \quad \text{and} \quad \int_1^2 \frac{dx}{x-1}$$

converges or diverges together

Now, the integral -

$$\int_1^2 \frac{dx}{x-1} = \lim_{a \rightarrow 1^+} \int_a^2 \frac{dx}{x-1}$$

$$= \lim_{a \rightarrow 1^+} \left[\ln(x-1) \right]_a^2 = \lim_{a \rightarrow 1^+} (\ln(1) - \ln(a-1))$$

$$= \lim_{a \rightarrow 1^+} (-\ln(a-1)) = +\infty$$

This diverges.

Conclusion - Since $\int_1^2 \frac{dx}{x-1}$ diverges and

$$\left(1 = \frac{x \ln x}{1-x} \sim \frac{1}{x-1} \right), \text{ and limit comparison Test}$$

$$\int_1^2 \frac{1}{\ln x} dx \text{ diverges.}$$

$$\lim_{x \rightarrow 1^+} \frac{\frac{1}{\ln x}}{\frac{1}{x-1}} = \lim_{x \rightarrow 1^+} \frac{x-1}{\ln x} = \lim_{x \rightarrow 1^+} \frac{1}{\frac{1}{x-1} \cdot \ln x}$$

therefore, limit comparison Test -

$$\lim_{x \rightarrow 1^+} \frac{1}{\ln x} = \infty$$

• Prove the inequality -

$$\frac{\pi^2}{9} < \int_{\pi/6}^{\pi/2} \frac{x}{\sin x} dx < \frac{2\pi^2}{9}$$

Sol.

Observe

~~sin~~ sine function,

on $[\pi/6, \pi/2]$

$$\frac{1}{2} \leq \sin x \leq 1 \quad \text{on } \left[\frac{\pi}{6}, \frac{\pi}{2}\right]$$

$$\Rightarrow 1 \leq \frac{1}{\sin x} \leq 2$$

$$x \in \left[\frac{\pi}{6}, \frac{\pi}{2}\right] \Rightarrow x > 0$$

multiply by 'x' in above inequality

$$x \leq \frac{x}{\sin x} \leq 2x$$

take the integration from $\frac{\pi}{6}$ to $\frac{\pi}{2}$.

$$\int_{\pi/6}^{\pi/2} x dx \leq \int_{\pi/6}^{\pi/2} \frac{x}{\sin x} dx \leq \int_{\pi/6}^{\pi/2} 2x dx$$

$$\Rightarrow \frac{x^2}{2} \Big|_{\pi/6}^{\pi/2} \leq \int_{\pi/6}^{\pi/2} \frac{x}{\sin x} dx \leq x^2 \Big|_{\pi/6}^{\pi/2}$$

Remarks-

$$\frac{1}{2} \left[\frac{\pi^2}{4} - \frac{\pi^2}{36} \right] \leq$$

- for this use $h(x) = \frac{x}{\sin x}$ is increasing. $f(x) \leq g(x) \Rightarrow \int_a^b f(x) \leq \int_a^b g(x)$
- (strictly increasing on $[\frac{\pi}{6}, \frac{\pi}{2}]$)
- This will give you strictness.
- you cannot use max. & min. of $f(x) = \frac{x}{\sin x}$
- $f(x) \leq g(x) \Rightarrow \int_a^b f(x) \leq \int_a^b g(x)$
- no strictness

Soln. (previous question).

Define $g(x) = \frac{x}{\sin x}$

show that $g(x)$ is strictly increasing on $[\frac{\pi}{6}, \frac{\pi}{2}]$

for this, $g'(x) = \frac{\sin x - x \cos x}{\sin^2 x}$

let $h(x) = \sin x - x \cos x$, $h(0) = 0$

and $h'(x) = x \sin x > 0$ for $x \in (0, \frac{\pi}{2})$

$\Rightarrow h(x)$ is strictly $\uparrow$ on $[0, \frac{\pi}{2}]$ with $h(0) = 0$

and $\sin^2 x > 0 \quad \forall x \in [\frac{\pi}{6}, \frac{\pi}{2}]$

Combining both, it gives — $\frac{x}{\sin x} \geq x$

$g'(x) = \frac{h(x)}{\sin^2 x} = \frac{\sin x - x \cos x}{\sin^2 x} > 0 \quad \forall x \in [\frac{\pi}{6}, \frac{\pi}{2}]$

imply $g(x) > 0 \quad \forall x \in [\frac{\pi}{6}, \frac{\pi}{2}]$

i.e. $g(x)$ is strictly increasing.

Now, bound $g(x)$ by its end point as it is strictly increasing,
so minimum & maximum will occur at end points —

$g(\frac{\pi}{6}) < g(x) < g(\frac{\pi}{2}) \quad \forall x, x \in (\frac{\pi}{6}, \frac{\pi}{2})$

and, $g(\frac{\pi}{6}) = \frac{4}{3}$

$g(\frac{\pi}{2}) = \frac{\pi}{2}$

So, we have our strictness -

$$\text{ie. } \frac{\pi}{3} < \frac{x}{\sin x} < \frac{\pi}{2} \text{ on } \left(\frac{\pi}{6}, \frac{\pi}{2}\right)$$

Now the length of the interval is $\frac{\pi}{2} - \frac{\pi}{6} = \frac{\pi}{3}$

taking integration w.r.t x with lower limit $\frac{\pi}{6}$ and upper limit $\frac{\pi}{2}$

$$\text{we get - } \int_{\pi/6}^{\pi/2} \frac{\pi}{2} dx < \int_{\pi/6}^{\pi/2} \frac{x}{\sin x} dx < \int_{\pi/6}^{\pi/2} \frac{\pi}{2} dx$$

$\Rightarrow$ Strictness will hold as $\frac{x}{\sin x}$ is strictly increasing function

$\rightarrow$ this is why you needed to show $\frac{x}{\sin x}$ is strictly $\uparrow$ fun.

So, our relation ~~became~~ after integration, becomes -

$$\frac{\pi}{3} \cdot \frac{\pi}{2} < \int_{\pi/6}^{\pi/2} \frac{x}{\sin x} dx < \frac{\pi}{3} \cdot \frac{\pi}{2}$$

$$\Rightarrow \frac{\pi^2}{9} < \int_{\pi/6}^{\pi/2} \frac{x}{\sin x} dx < \frac{\pi^2}{6}$$

$$\text{and } \frac{\pi^2}{6} < \frac{2\pi^2}{9}$$

$$\text{So, conclusion - } \left[\frac{\pi^2}{9} < \int_{\pi/6}^{\pi/2} \frac{x}{\sin x} dx < \frac{2\pi^2}{9} \right]$$

- Find the range of $p (>0)$ for which the series -

$$\frac{1}{(1+a)^p} - \frac{1}{(2+a)^p} + \frac{1}{(3+a)^p} - \dots, \quad a > 0 \text{ is -}$$

- absolutely convergent
- conditionally convst.

Soln.

~~Define~~

Define $s := \sum_{n=1}^{\infty} (-1)^{n+1} u_n$

where $u_n = \frac{1}{(n+a)^p}, \quad a > 0, p > 0$

- Abs. convgt.

Absolutely convergent requires -

$$\sum_{n=1}^{\infty} |(-1)^{n+1} u_n| = \sum_{n=1}^{\infty} \frac{1}{(n+a)^p} < \infty$$

Since 'a' is fixed, $\frac{1}{(n+a)^p} \sim \frac{1}{n^p}$ as $n \rightarrow \infty$

(compare it via limit comparison test) -

- state the test properly -

$$\lim_{n \rightarrow \infty} \frac{\frac{1}{(n+a)^p}}{\frac{1}{n^p}} = \lim_{n \rightarrow \infty} \left(\frac{n}{n+a} \right)^p = 1 \neq 0$$

finite

this imply $\sum_{n=1}^{\infty} \frac{1}{(n+a)^p}$ convgt. iff $\sum_{n=1}^{\infty} \frac{1}{n^p}$ convgt.

and p-series converges for $p > 1$

$\boxed{\text{S converges absolutely iff } p > 1}$

• Conditionally convergent -

For $b \leq 1$, series diverges absolutely.

For ordinary convergence, we test it using Leibniz Alternating series Test -

$$u_n = \frac{1}{(n+a)^b}$$

• state the statement -

• +ve clearly $u_n > 0 \forall n > 0$ as $n+a > 0$

• monotonically decreasing.

Define $\phi(x) = \frac{1}{(x+a)^b}$ has -

$$\phi'(x) = -\frac{b}{(x+a)^{b+1}} < 0 \text{ for } x > -a$$

so ϕ is strictly decreasing for all $x > 0$,

in particular $u_{n+1} < u_n \forall n$.

• Now, the limit,

$$\lim_{n \rightarrow \infty} u_n = \lim_{n \rightarrow \infty} \frac{1}{(n+a)^b} = 0 \text{ since } b > 0$$

All the three conditions hold for every $b > 0$.

so it converges for all $b > 0$.

[• absolutely convergent : $b > 1$
• conditionally convergent : $0 < b \leq 1$]

- show that if a function f defined on an open interval $(a, b) \subseteq \mathbb{R}$ is convex, then f is continuous.
- Now by counterexample, if the condition of open interval is dropped, then the convex function need not be continuous.

Ex

Solⁿ

CORE IDEA:- A convex function's slope are monotonic (the three-chord/slope inequality) which lets you sandwich local behaviours between two fixed bounding slope, it gives a local lipschitz bound, hence continuity at every interior point.

This is why openness matters.

If you want more - The sandwich argument needs point on both sides of x within the domain. At an end point of a closed interval, you only have one side, so the bound breaks down, this allows a jump discontinuity there.

→ let $f(x): (a, b) \rightarrow \mathbb{R}$ be convex, meaning for all $x, y \in (a, b)$ and $\lambda \in [0, 1]$,

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y)$$

To show - $f(x)$ is continuous at every point $x_0 \in (a, b)$.

Three-chord (slope) inequality -

Claim - If $u < v < w$ all lie in (a, b) , then -

$$\frac{f(v) - f(u)}{v - u} \leq \frac{f(w) - f(u)}{w - u} \leq \frac{f(w) - f(v)}{w - v}$$

Let $v = \lambda u + (1-\lambda)w$ where $\lambda = \frac{w-v}{w-u} \in (0,1)$

Since $f(x)$ is continuous

$$\Rightarrow f(v) \leq \lambda f(u) + (1-\lambda)f(w)$$

Subtracting $f(u)$ from both sides and using $1-\lambda = \frac{v-u}{w-u}$

$$f(v) - f(u) \leq (1-\lambda)(f(w) - f(u)) = \frac{v-u}{w-u}(f(w) - f(u))$$

$$\Rightarrow \frac{f(v) - f(u)}{v - u} \leq \frac{f(w) - f(u)}{w - u}$$

Similarly, we can have -

$$\frac{f(w) - f(u)}{w - u} \leq \frac{f(w) - f(v)}{w - v}$$

It shows that the slope function $x \mapsto \frac{f(x) - f(x_0)}{x - x_0}$ is non-decreasing in x (for fixed x_0) & $x \neq x_0$ in (a,b) .

Now, we have to show the Bound of $f(x)$ near x_0 , using nearby points -
and we will use this to show the continuity -

Fix, $x_0 \in (a,b)$. Choose $u, w \in (a,b)$ with -

$$a < u < x_0 < w < b$$

(Why the existence of u and w ?) bcz (a,b) is open, so x_0 has room on both sides.

For $x \in (u,w)$, $x \neq x_0$. apply three-chord inequality -

• If $x_0 < x < w$, using $u < x_0 < x$ then $x_0 < x < w$

$$\Rightarrow \frac{f(x_0) - f(u)}{x_0 - u} \leq \frac{f(x) - f(x_0)}{x - x_0} \leq \frac{f(w) - f(x_0)}{w - x_0}$$

• If $u < x < x_0$, similarly—

$$\frac{f(x_0) - f(u)}{x_0 - u} \leq \frac{f(x) - f(x_0)}{x - x_0} \leq \frac{f(w) - f(x_0)}{w - x_0}$$

In both the cases $\frac{f(x) - f(x_0)}{x - x_0}$ is trapped b/w the two real numbers

$$\text{Define } m' = \frac{f(x_0) - f(u)}{x_0 - u}, \quad M' = \frac{f(w) - f(x_0)}{w - x_0}$$

which depend only on u, w, x_0 (not on x). let $K = \max(|m'|, |M'|)$

Now, the continuity part—

For all $x \in (u, w) \setminus \{x_0\} = (u, x_0) \cup (x_0, w)$

$$\left| \frac{f(x) - f(x_0)}{x - x_0} \right| \leq K \Rightarrow |f(x) - f(x_0)| \leq K|x - x_0|$$

Hence, for any $\varepsilon > 0$, choosing $\delta = \min(x_0 - u, w - x_0, \frac{\varepsilon}{K+1})$

$$|f(x) - f(x_0)| < \varepsilon \text{ whenever } |x - x_0| < \delta.$$

$\Rightarrow f$ is continuous at x_0 .

Since x_0 was arbitrary $\Rightarrow f$ is continuous on (u, w) .

[OR you can show that f is Lipschitz $\Rightarrow$ continuous].

Counter Example -

Think before proceeding further -
(use core idea)

$$\text{let } [a, b] = [0, 1]$$

$$\text{define } f(x) := \begin{cases} 0 & 0 \leq x < 1 \\ 1 & x = 1 \end{cases}$$

→ check that it is convex using standard definition and show that there is a ~~cont~~ discontinuity at $x=1$.

• Suppose $\mathbb{R}$ be the set of all real numbers and $f: \mathbb{R} \rightarrow \mathbb{R}$ is a function such that the following conditions hold for all $x, y \in \mathbb{R}$.

1. $f(x+y) = f(x) + f(y)$

2. $f(xy) = f(x)f(y)$

Show that $\forall x \in \mathbb{R}$ either $f(x) = 0$ or $f(x) = x$. $\forall x \in \mathbb{R}$

Soln.

$f: \mathbb{R} \rightarrow \mathbb{R}$, is ~~both additive and multiplicative fun.~~

i.e.

$$f(x+y) = f(x) + f(y)$$

$$f(xy) = f(x)f(y)$$

$$\forall x, y \in \mathbb{R}.$$

Basic Values—

$$f(0) = f(0+0) = f(0) + f(0) \quad \text{by property.}$$

$$\Rightarrow f(0) = f(0) + f(0)$$

$$\Rightarrow f(0) = 0 \quad \text{--- ①}$$

$$f(1) = f(1 \cdot 1) = (f(1))^2 \Rightarrow f(1)(f(1) - 1) = 0$$

$$\Rightarrow f(1) = 0 \quad \text{or} \quad f(1) = 1$$

$$\text{i.e.} \quad f(1) \in \{0, 1\}$$

$$\begin{aligned} &\bullet \text{ show} \\ &f(x) = -f(x) \\ &\forall x \in \mathbb{R}. \end{aligned}$$

Hint-

$$\begin{aligned} f(x-x) &= f(0) \\ &= f(x) + f(-x) \end{aligned}$$

Case-I - When $f(1) = 0$

$$\forall x \in \mathbb{R}, \quad f(x) = f(x \cdot 1) = f(x) \cdot f(1) = f(x) \cdot 0$$

$$\Rightarrow \cancel{f(x)} \quad f(x) = 0 \quad \forall x \in \mathbb{R}.$$

Case-II - When $f(1) = 1$, To show $f(x) = x, \forall x \in \mathbb{R}$.

$$\text{Observe,} \quad f(nx) = f(n) \cdot f(x) \quad \text{by multiplicativity} \quad x \in \mathbb{R}, n \in \mathbb{N}$$

$$\text{Now,} \quad f(n) = \underbrace{f(1+1+\dots+1)}_{n\text{-times}} = n \cdot f(1) = n$$

$$\text{this implies} \quad f(nx) = n \cdot f(x)$$

$$\text{similarity for ration } \frac{p}{q}, \quad q \in \mathbb{N}, \quad x = \frac{p}{q} \text{ let } n = q$$

$$f(nx) = n \cdot f(x) = n \cdot f\left(\frac{p}{q}\right)$$

$$\Rightarrow f\left(\frac{b}{2}\right) = \frac{b}{2}$$

$$\Rightarrow f(r) = r \quad \forall r \in \mathbb{Q}$$

Note- You cannot use sequential criterion to show $f(x)=x \quad \forall x \in \mathbb{R}$ as the given function is not continuous.

• There is a question, 'f' is additive continuous, in this particular question, you can use sequential criterion

Back to the question -

- f is non-negative on $[0, \infty)$

$$\text{for any } x \geq 0, \quad x = (\sqrt{x})^2$$

$$\Rightarrow f(x) = f((\sqrt{x})^2) = (f(\sqrt{x}))^2 \geq 0$$

- f is monotonically increasing.

$$\text{let } x \leq y, \text{ then } y - x \geq 0$$

$$f(y) = f(x + (y-x)) = f(x) + f(y-x) \geq f(x)$$

So, f is non-decreasing on $\mathbb{R}$.

- Now, squeeze with rationals to get $f(x) = x \quad \forall x \in \mathbb{R}$.

Fix $x \in \mathbb{R}$, and $\mathbb{Q}$ is dense in $\mathbb{R}$ i.e. $\bar{\mathbb{Q}} = \mathbb{R}$

for any given $\epsilon > 0$, choose rationals $\frac{b}{2}$ with

$$x - \epsilon < b \leq x \leq 2 < x + \epsilon$$

By monotonicity

$$p = f(b) \leq f(x) \leq f(a) = 2$$

$$x - \epsilon < f(x) < x + \epsilon$$

$$\Rightarrow |f(x) - x| < \epsilon$$

Use property here -

$\epsilon > 0$, and if for any $\epsilon > 0$, ~~$0 < \epsilon < \epsilon$~~

$$0 < \epsilon < \epsilon \Rightarrow \underline{a = 0}$$

implies $f(x) = x \quad \forall x \in \mathbb{R}$.

- let $x_1 = 2$ and $x_{n+1} = \sqrt{x_n + 2}$, $n = 1, 2, \dots$. show that the sequence $x_1, x_2, x_3, \dots$ is convergent.

Sol.

We will show $\{x_n\}$ is monotonically increasing and bounded above, hence convergent by the Monotone Convergence Theorem.

Boundedness -

Claim - $2 \leq x_n < 5 \quad \forall n \in \mathbb{N}$

prove using mathematical induction -

$x_1 = 2$ clearly true.

suppose it is true for 'n'. i.e. $2 \leq x_n < 5$

we will show that it holds for $n+1$:

By previous assumptions - $2 \leq x_n < 5$

$$\Rightarrow x_n + 20 < 25$$

$$\Rightarrow \sqrt{x_n + 20} < \sqrt{25}$$

$$\Rightarrow x_{n+1} < 5$$

$$\text{Also, } x_n \geq 2 > 0 \Rightarrow x_n + 20 > 0 \Rightarrow x_{n+1} = \sqrt{x_n + 20} \geq \sqrt{20} > 2$$

$$\text{i.e. } x_{n+1} \geq 2$$

Combining, $2 \leq x_n < 5 \quad \forall n \in \mathbb{N}$.

• Now, monotonically increasing, claim $x_{n+1} > x_n \quad \forall n$.

$$\begin{aligned} \text{Observe - } x_{n+2} - x_{n+1} &= \sqrt{x_{n+1} + 20} - \sqrt{x_n + 20} \\ &= \frac{x_{n+1} - x_n}{\sqrt{x_{n+1} + 20} + \sqrt{x_n + 20}} = \frac{x_{n+1} - x_n}{x_{n+2} + x_{n+1}} \end{aligned}$$

Since $x_n \geq 2 \quad \forall n \in \mathbb{N}$. and $x_n > 0 \quad \forall n$

imply $x_{n+2} - x_{n+1}$ has the same sign as $x_{n+1} - x_n$.

$$\text{i.e. } x_2 - x_1 > 0 \Rightarrow x_3 - x_2 > 0 \text{ and so on...}$$

$$\text{implies } x_{n+1} - x_n > 0 \quad \forall n \geq 1.$$

i.e. $\{x_n\}$ is strictly increasing sequence.

$\{x_n\}$ is increasing and bounded above by 5, so by the

Monotone Convergence Theorem -

$\{x_n\}$ converges to its supremum. (state it properly).
(say L)

$$x_{n+1} = \sqrt{x_n + 20}$$

Taking $n \rightarrow \infty$ and using continuity
of square root function -

$$L = \sqrt{L + 20}$$

$$\Rightarrow L^2 - L - 20 = 0 \Rightarrow L = -4, 5.$$

$$\text{as } x_n > 0 \Rightarrow \boxed{L = 5}$$

$$\text{i.e. } \lim_{n \rightarrow \infty} x_n = 5.$$

Find the supremum and infimum of $\frac{x}{\sin x}$ on $(0, \frac{\pi}{2}]$

$$\text{Define } g(x) := \frac{x}{\sin x} \quad \text{on } (0, \frac{\pi}{2}]$$

$$\text{Claim-} \quad \inf_{(0, \frac{\pi}{2}]} g(x) = 1, \quad \sup_{(0, \frac{\pi}{2}]} g(x) = \frac{\pi}{2}$$

first show that g is strictly increasing on $[0, \pi/2]$

(We did this 5 questions back; please check that)

Now, since g is strictly increasing on $[0, \pi/2]$

the supremum will take place at end point (largest one)

i.e. at $x = \frac{\pi}{2}$.

$$g(x) \leq g\left(\frac{\pi}{2}\right) = \frac{\pi}{2} \quad \forall x \in [0, \pi/2]$$

i.e. $\sup_{[0, \pi/2]} g(x) = \frac{\pi}{2} = \max_{[0, \pi/2]} g(x)$ (maximum attained)

• Infimum -

$$\lim_{x \rightarrow 0^+} g(x) = \lim_{x \rightarrow 0^+} \frac{x}{\sin x} = \lim_{x \rightarrow 0^+} \frac{1}{\frac{\sin x}{x}} = \frac{1}{\lim_{x \rightarrow 0^+} \frac{\sin x}{x}} = 1$$

$$\left[\text{As } \lim_{x \rightarrow 0^+} \frac{\sin x}{x} = 1 \right]$$

imply $\frac{x}{\sin x}$ is bounded below by 1.

now, show that it is the greatest lower bound -

for any $\epsilon > 0$, since $\lim_{x \rightarrow 0^+} g(x) = 1$, there exist $x \in [0, \pi/2]$

with $g(x) < 1 + \epsilon$.

so, $\inf_{[0, \pi/2]} \frac{x}{\sin x} = 1$ (not attained on $[0, \pi/2]$)

- let $f(t) = \int_0^t [x] dy$

where $[x]$ denotes the largest integer less than or equal to x .

1. Determine all points at which f is differentiable.
2. same for ' f ' is continuous but not differentiable.

Solⁿ.

$$f(t) = \int_0^t [x] dx$$

$[x]$ is largest integer less than or equal to x .

$[x]$ is bounded on any finite interval and Riemann Integrable (it has only finitely many jump discontinuities on any bounded interval).

- Claim- f is continuous everywhere -

for any $x, y \in \mathbb{R}$. Since $[x] \leq \max(|x|, |y|+1) = B$

on $[x, y]$ or $[y, x]$

$$\cancel{f(x) - f(y)} \quad |f(x) - f(y)| = \left| \int_y^x [x] dx \right| \leq B|x-y|$$

So, f is Lipschitz (locally), hence continuous on all of $\mathbb{R}$.

As $x, y \in \mathbb{R}$ were random.

imply - $f(x)$ is continuous everywhere.

• Differentiability -

For ~~$x \in \mathbb{R}$~~ let $t, x \in \mathbb{R}$ be any random real numbers.

$$\text{For } \cancel{x \in \mathbb{R}} \quad n \leq t < n+1 \quad (n \in \mathbb{Z})$$

$$f(t) = \int_0^n [x] dx + \int_n^t n \cdot dx = f(n) + n(t-n)$$

where $f(n) = \sum_{k=0}^{n-1} k$, ($n \geq 0$) or the corresponding negative sum for $n < 0$,

either way $f(n)$ is fixed constant on $[n, n+1)$,

$$f(t) = f(n) + n(t-n) \Rightarrow f'(t) = n = [t] \quad \forall t \in (n, n+1)$$

so, f is differentiable at every non-integer t .

and

$$f'(t) = [x]$$

†

• Non-differentiability at integers -

We will do this by RHL and LHL of derivative at integers.

Fix $t = n \in \mathbb{Z}$

Right derivative - on $[n, n+1)$ $f(t) = f(n) + n(t-n)$

$$f'(n+) = \lim_{t \rightarrow n+} \frac{f(t) - f(n)}{t - n} = n$$

Left derivative - on $[n-1, n)$, $f(t) = f(n-1) + (n-1)(t - (n-1))$

$$\text{and } f(n) = f(n-1) + (n-1)$$

$$f'(n-) = \lim_{t \rightarrow n-} \frac{f(t) - f(n)}{t - n} = n-1$$

Since both limits are unequal and differ by 1.

So, f is not differentiable at any integer.

Conclusion.

• f is continuous on all of $\mathbb{R}$

• differentiable everywhere except at integers.

• Two sequences $\{x_n\}$ and $\{y_n\}$ are defined inductively by the following:

$$x_1 = \frac{1}{2}, y_1 = 1 \text{ and } x_n = \sqrt{x_{n-1} \cdot y_{n-1}}, n \geq 2.$$

$$\text{and } y_n = \frac{1}{2} \left(\frac{1}{x_n} + \frac{1}{y_{n-1}} \right) \quad n \geq 2$$

prove that $x_{n-1} < x_n < y_n < y_{n-1}$. $n \geq 2$.

and deduce that both the sequences converge to the same limit L , where $\frac{1}{2} < L < 1$.

Solⁿ.

Observe- $\{x_n\}$ is the geometric mean of x_{n-1} and y_{n-1}

and $\{y_n\}$ is the harmonic mean of x_n and y_{n-1}

~~Claim~~ Fact - geometric mean and harmonic mean of two numbers lies strictly between the two numbers.

i.e. if $0 < p < q$ then $p < \sqrt{pq} < q$ and $p < \frac{2pq}{p+q} < q$

• Claim - $x_{n-1} < x_n < y_n < y_{n-1}$ by induction -

Base Case - $n=2$. $x_1 = \frac{1}{2} < 1 = y_1$

To prove our claim for base case, carefully use stated fact.

$$x_1 < \sqrt{x_1 y_1} = x_2 < y_1$$

(You can prove the fact, as it is much easier.)

Since $x_2 < y_1$, apply fact here -

$$x_2 < y_2 = \frac{2x_2 y_1}{x_2 + y_1} < y_1$$

Combining $x_1 < x_2 < y_2 < y_1$

Inductive Step - Suppose $x_{n-2} < x_{n-1} < y_{n-1} < y_{n-2}$ holds.

In particular $x_{n-1} < y_{n-1}$. So By the fact

letting $p = x_{n-1}$, $q = y_{n-1}$

$$x_{n-1} < x_n = \sqrt{x_{n-1} y_{n-1}} < y_{n-1}$$

Since $x_n < y_{n-1}$, apply fact again ($p = x_n$, $q = y_{n-1}$)

$$\text{So, } x_n < y_n = \frac{2x_n y_{n-1}}{x_n + y_{n-1}} < y_{n-1}$$

Combining - $x_{n-1} < x_n < y_n < y_{n-1}$ $\forall n \geq 2$

• Monotonicity and boundedness-

From the chain $x_{n-1} < x_n < y_n < y_{n+1}$, $n \geq 2$

$\{x_n\}$ is strictly increasing and

$\{y_n\}$ is strictly decreasing

i.e. $x_1 < x_2 < \dots < x_n < y_n < \cancel{y_{n+1}} < y_{n-2} < \dots < y_2 < y_1$

So, by monotone convergence theorem

$x_n \rightarrow l_1$ and $y_n \rightarrow l_2$ with $l_1 \leq l_2$

• show $l_1 = l_2$.

let $d_n = y_n - x_n \geq 0$

$$d_n = y_n - x_n = \frac{2x_n y_{n-1}}{x_n + y_n} - x_n = \frac{x_n (y_{n-1} - x_n)}{x_n + y_{n-1}}$$

Carefully observe-

$x_{n-1} < x_n$, we have

~~$y_{n-1} < y_n$~~

$$y_{n-1} - x_n < y_{n-1} - x_{n-1} = d_{n-1}$$

Since $x_n < y_{n-1}$, we have-

$$x_n + y_{n-1} > 2x_n \quad \text{so, } \frac{x_n}{x_n + y_{n-1}} < \frac{1}{2}$$

$$\text{Combining } d_n < \frac{1}{2} d_{n-1} \quad \forall n \geq 2$$

Since $d_1 = y_1 - x_1 = \frac{1}{2}$, iteration gives -

$$0 \leq d_n < \left(\frac{1}{2}\right)^{n-1} d_1 = \left(\frac{1}{2}\right)^{n-1} \rightarrow 0$$

So, by squeeze theorem, $\lim_{n \rightarrow \infty} d_n = 0$

i.e. $\lim_{n \rightarrow \infty} (y_n - x_n) = 0$

both limit exists

imply $\lim_{n \rightarrow \infty} y_n - \lim_{n \rightarrow \infty} x_n = 0$

$$\Rightarrow \left[\lim_{n \rightarrow \infty} x_n = \lim_{n \rightarrow \infty} y_n \right] = (\text{say } l)$$

Bound the limit -

Since $\{x_n\}$ is strictly increasing with $x_n \rightarrow l$

$$\cancel{x_1} \quad x_1 < x_2 \leq l \Rightarrow \frac{1}{2} = x_1 < l$$

Since $\{y_n\}$ is strictly decreasing with $y_n \rightarrow l$

$$l \leq y_2 < y_1 \Rightarrow l < 1 = y_1$$

Conclusion.

$x_n \uparrow l$ (increasing and converge to l)

$y_n \downarrow l$ (decreasing and converging to l)

and $\left[\frac{1}{2} < l < 1 \right]$

- For the function - $f: (0, \infty) \rightarrow \mathbb{R}$, given by -

$$f(x) = x^2 \sin \frac{1}{x}, \quad 0 < x < \infty$$

show that there is a differentiable function $g: \mathbb{R} \rightarrow \mathbb{R}$ that extends f .

Solⁿ We have to find/construct a differentiable which extends f on whole real numbers.

Define $g: \mathbb{R} \rightarrow \mathbb{R}$ by -

$$g(x) = \begin{cases} x^2 \sin \frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

For $x > 0$, $g(x) = f(x)$, so g extends f .

We will show that this extension i.e. $g(x)$ is differentiable on all of $\mathbb{R}$.

- Differentiability at $x \neq 0$, $x \in \mathbb{R}$.

$$g'(x) = 2x \sin \frac{1}{x} + x^2 \cos \frac{1}{x} \cdot \left(-\frac{1}{x^2}\right) = 2x \sin \frac{1}{x} - \cos \frac{1}{x}, \quad x \neq 0$$

this is a well-defined function for $x \neq 0$, no ambiguity here.

so, g is differentiable at every $x \neq 0$.

- Differentiability at $x=0$.

$$g'(0) = \lim_{h \rightarrow 0} \frac{g(h) - g(0)}{h - 0} = \lim_{h \rightarrow 0} \frac{h^2 \sin \frac{1}{h} - 0}{h - 0} = \lim_{h \rightarrow 0} h \sin \frac{1}{h}$$

Since $|\sin \frac{1}{h}| \leq 1 \quad \forall h \neq 0$, we have -

$$0 \leq \left| h \sin \frac{1}{h} \right| \leq |h|$$

$$\text{and} \quad \lim_{h \rightarrow 0} 0 \leq \lim_{h \rightarrow 0} h \sin \frac{1}{h} \leq \lim_{h \rightarrow 0} |h| \rightarrow 0$$

$$\Rightarrow \quad \lim_{h \rightarrow 0} h \sin \frac{1}{h} = 0$$

Therefore $g'(0)$ exists and $g'(0) = 0$.

Conclusion - g is differentiable at every $x \neq 0$ and at $x=0$,
so, g is diff. on all of $\mathbb{R}$.

$$g'(x) = \begin{cases} 2x \sin \frac{1}{x} - \cos \frac{1}{x} & , x \neq 0 \\ 0 & , x = 0 \end{cases}$$

and restriction of $g(x)$ on $(0, \infty)$ is $f(x)$, i.e. $g|_{(0, \infty)} = f$

so, g is a differentiable extension of f on $\mathbb{R}$.

Bonus - This $g(x)$ is an example of a diff. funⁿ whose derivative is not conti.

• let $f: \mathbb{R} \rightarrow \mathbb{R}$ be a continuous function such that

$\lim_{x \rightarrow \infty} f(x)$ and $\lim_{x \rightarrow -\infty} f(x)$ exist and are finite.

Prove that f is uniformly continuous on $\mathbb{R}$.

Solⁿ let $f: \mathbb{R} \rightarrow \mathbb{R}$ be continuous, with—

$$\lim_{x \rightarrow +\infty} f(x) = L_2 \in \mathbb{R} \quad \text{and} \quad \lim_{x \rightarrow -\infty} f(x) = L_1 \in \mathbb{R}$$

We will show that— for every $\varepsilon > 0$, there exist $\delta > 0$ such that

$$\forall x, y \in \mathbb{R}$$

$$|x - y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon$$

Now, we have to control the functional value as value of x when $x \rightarrow \infty$.

Fix $\varepsilon > 0$, since $\lim_{x \rightarrow +\infty} f(x) = L_2$, there exist $M_2 > 0$, $M_2 \in \mathbb{R}$

such that — $x > M_2$ (way bigger than M_2)
— a larger number.

$$\text{implies } |f(x) - L_2| < \varepsilon/4$$

~~As~~, Consequently, for $x, y > M_2$

$$|f(x) - f(y)| \leq |f(x) - L_2| + |L_2 - f(y)| < \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \frac{\varepsilon}{2}$$

Similarly, since $\lim_{x \rightarrow -\infty} f(x) = L_2$, there exist $M_1 > 0, M_1 \in \mathbb{R}$ such that

$$\cancel{x \leq -M_2} \quad x < -M_2 \Rightarrow |f(x) - L_2| < \epsilon/4$$

hence for $x, y < -M_1$, we have -

$$|f(x) - f(y)| < \frac{\epsilon}{2} \quad (\text{by same process})$$

Now,

Define $M := \max(M_1, M_2)$

combining both the results - we have

$$|f(x) - f(y)| < \frac{\epsilon}{2} \quad \forall \quad x, y > M \quad \text{and} \quad \forall \quad x, y < -M$$

Now, we only left with real numbers inside compact (close and bounded) in $[-M-1, M+1]$.

As f is continuous and interval is close and bounded.

f is uniformly continuous. ~~(by Heine theorem)~~
on $[-M-1, M+1]$ (by Uniform Continuity theorem.)

So, there exist $\delta > 0$ such that

$$\forall x, y \in [-M-1, M+1],$$

$$|f(x) - f(y)| < \frac{\epsilon}{2} \quad \text{whenever} \quad |x - y| < \delta$$

Conclusion - for every $\epsilon > 0$, we have $\delta > 0$ (independent of x & y) such that

$$|x - y| < \delta \Rightarrow |f(x) - f(y)| < \epsilon \quad \forall x, y \in \mathbb{R}.$$

Hence f is uniformly continuous on $\mathbb{R}$.