

Shreya

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REAL ANALYSIS TEST 1

Instructions: Questions 1-4 each contain two parts. Attempt any **three** questions out of the first four, solving one part from each. Question 5 is compulsory. Each question carries 15 marks. Total marks: 60.

Revised marks

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1. Show that if a convergent series contains only finitely many negative terms, then it is absolutely convergent.
OR

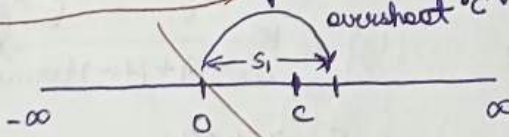
Show that if a series is conditionally convergent, the series of its positive terms diverges, and the series of its negative terms diverges.

Let $\sum a_n$ (say) be conditionally convergent, Then $\sum a_n$ converges and $\sum |a_n| = \sum a_{n+} + \sum a_{n-}$ diverges, where $a_{n+} > 0$ are positive terms and a_{n-} are negative terms.

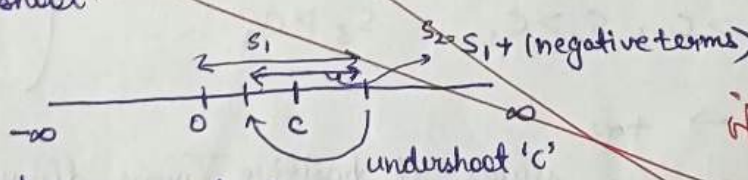
Since, $\sum a_n$ converges, $a_n \rightarrow 0$

Let $c \in \mathbb{R}$ be arbitrary

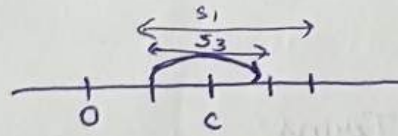
- Add positive terms until the partial sum first exceeds c . This is an overshoot



- Now add negative terms until the partial sum first falls below c . This is an undershoot.



- Again add positive terms until it overshoots c :



- Again add negative terms until it undershoots c .

$S_4 < c$

Thus, $S_1 > c > S_2, S_3 > c > S_4, S_5 > c > S_6 \dots$

Since $\sum a_n$ converges,

$a_n \rightarrow 0$ & $\sum |a_n| = \infty$

its epsilon
First show
the existence
of $S_1 \in \mathbb{R}$
such that
 $S_1 > c$

Define $n \in \mathbb{N}$
notation is a_{n+} / a_{n-}

give reason or
justification, how can
you keep adding
+ve term and -ve term

Case I - Positive terms.

Let $c > 0$ be arbitrary $(c \in \mathbb{R})$

Start adding the positive terms of the series
since, $\sum |a_n| = \infty$ (divergent)

the positive terms cannot remain bounded as the $\sum a_n$ converges

Hence the positive partial sums can be made to exceed any prescribed c .

Hence, choose positive terms until the partial sum first exceeds c :

$$S_1 > c$$

Now choose a larger c , c_2 (say) $> c$, and continue adding positive terms until

$$S_2 > c_2$$

Similarly, choose

$$c < c_2 < c_3 < \dots, c_n \rightarrow +\infty$$

Then ~~the~~ positive partial sums satisfy

$$S_1 > c, S_2 > c_2, S_3 > c_3, \dots$$

$$S_0,$$

$$S_n \rightarrow +\infty$$

Hence, the series of positive terms diverges to $+\infty$

$$\sum a_n^+ = +\infty$$

Case II - Negative Terms.

Let $c < 0$ be arbitrary

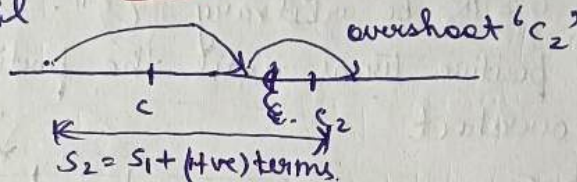
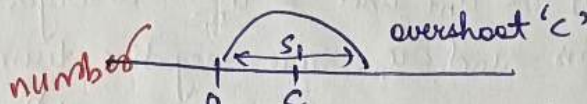
Start adding the ~~the~~ negative terms of the series

since, $\sum |a_n| = \infty$ (divergent)

the negative terms cannot remain bounded as the $\sum a_n$ converges

Hence the partial sum will eventually fall below c .

this
region is
covered
in your first
half relation



Relⁿ b/w
 S_1, S_2, S_3 ?

Great Explanation -

✗

But, the stages that you have chosen to select $s_1, s_2, s_3 \dots$

meaning the value of $c_i, i \geq 1$

Just fix them by taking them as the natural numbers.

i.e. $c_1 = 1, c_2 = 2, \dots$

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for explaining well.

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sequence using ~~sequence~~ numbers, which diverges to ∞ and this is the crux of reasoning. You have to make a natural

There is another method - (see Nikhil's solution).

2. If $\{a_n\}$ is a decreasing sequence of strictly positive numbers and $\sum a_n$ converges, show $\lim_n na_n = 0$. Give an example of a divergent series with $\{a_n\}$ decreasing and $\lim na_n = 0$.

OR

→ If $\{a_n\}$ is a sequence and $\lim n^2 a_n$ exists in $\mathbb{R}$, show $\sum a_n$ is absolutely convergent.

Let $\lim_{n \rightarrow \infty} n^2 a_n = L \quad (L \in \mathbb{R})$

Here ~~$\{a_n\} = n^2 a_n$~~ $X_n = n^2 a_n$

~~$\forall \epsilon > 0, \exists \delta > 0$~~ Using the definition of limit of a sequence

$\forall \epsilon > 0, \exists m(\epsilon) \in \mathbb{N} \nexists |X_n - L| < \epsilon \quad \forall n \geq m(\epsilon)$

Let $\epsilon = 1$, then

$\epsilon \rightarrow$ belongs to
 $\epsilon \rightarrow$ Epsilon

in particular,

$|n^2 a_n - L| < 1$

$|n^2 a_n| - |L| \leq 1$

$|n^2 a_n| < |L| + 1$

$n^2 |a_n| < |L| + 1$

$\Rightarrow |a_n| < \frac{|L| + 1}{n^2}$

$\forall n \geq m(\epsilon)$

define
[By triangle inequality]

$||a| - |b|| \leq |a \pm b| \leq |a| + |b|$

Can use
Comparison Test
let $\sum a_n = \frac{1}{n^2}$

and use it directly

Using comparison test for absolute convergent series. which states that if $\lim_{n \rightarrow \infty} \left| \frac{x_n}{y_n} \right|$ exists finitely

Here $|L|$ is fixed real number, let $m = |L| + 1$ be a positive constant.

$0 \leq |a_n| \leq \frac{m}{n^2} \quad \forall n \geq m(\epsilon)$

$\sum a_n = \sum \frac{m}{n^2} = m \sum \frac{1}{n^2}$

Now we have to check the convergence of $\sum \frac{1}{n^2}$

~~$\frac{1}{16}$~~ ~~$\frac{7}{15}$~~
converges

Let $X_n = \frac{1}{n^p}$

$$\left| \frac{X_{n+1}}{X_n} \right| = \frac{1}{(n+1)^p} \cdot \frac{n^p}{1} = \frac{n^p}{(n+1)^p} = \left(\frac{n}{n+1} \right)^p = \left(\frac{1}{1+\frac{1}{n}} \right)^p$$

$$= \left(1 + \frac{1}{n} \right)^{-p}$$

Using Raabe's Test,

$$\sigma = \lim_{n \rightarrow \infty} n \left[1 - \left(1 + \frac{1}{n} \right)^{-p} \right]$$

$$= \lim_{n \rightarrow \infty} n \left[1 - \left(1 - \frac{p}{n} + \frac{p(p+1)}{2n^2} - \dots \right) \right]$$

$$= \lim_{n \rightarrow \infty} n \left[\frac{p}{n} - \frac{p(p+1)}{2n^2} + \dots \right]$$

$$= \lim_{n \rightarrow \infty} n \cdot \frac{p}{n} - \lim_{n \rightarrow \infty} n \cdot \frac{p(p+1)}{2n^2} + \dots$$

$$= p - \lim_{n \rightarrow \infty} \frac{p(p+1)}{2n} + \dots$$

$$= p$$

- taking limit n tends to infinity
- p is fixed

Convergence of Raabe's Test -
When $\sigma > 1$

$$\frac{p(p+1)}{M \cdot n^q} \rightarrow 0$$

$$q \in \mathbb{N}$$

$$M < \infty$$

i.e. $p > 1$

So, $\sum X_n = \sum \frac{1}{n^p}$ converges for $p > 1$

Hence, $\sum a_n$ converges As $\sum \frac{1}{n^2}$ converges.

3. Let (s_n) be a sequence of nonnegative numbers, and for each n define

$$\sigma_n = \frac{1}{n}(s_1 + s_2 + \dots + s_n)$$

- (a) Show that if $\lim s_n$ exists, then $\lim \sigma_n$ exists and $\lim \sigma_n = \lim s_n$.
 (b) Give an example where $\lim \sigma_n$ exists, but $\lim s_n$ does not exist.

OR

Let $S = \left\{ \frac{1}{2^m} + \frac{1}{2^n} : m, n \in \mathbb{N} \right\}$

- (a) Show that 0 is a limit point of S .
 (b) If $k \in \mathbb{N}$, show that $\frac{1}{2^k}$ is a limit point of S .
 (c) Find S' (the derived set of S).

Let $S = \left\{ \frac{1}{2^m} + \frac{1}{2^n} : m, n \in \mathbb{N} \right\}$

a) Let $m=n$,

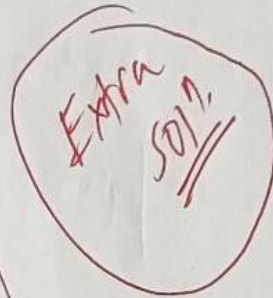
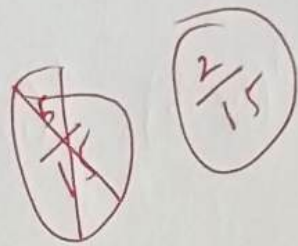
$$S_n = \frac{1}{2^n} + \frac{1}{2^n} = \frac{2}{2^n} = \frac{1}{2^{n-1}}$$

$$\text{Now } \lim_{n \rightarrow \infty} S_n = \lim_{n \rightarrow \infty} \frac{1}{2^{n-1}} = 0$$

Here, $S_n \in S \forall n$ & $S_n \rightarrow 0$, so 0 is a limit point of S .

b) If $k \in \mathbb{N}$

$$\lim_{n \rightarrow \infty} \left(\frac{1}{2^k} + \frac{1}{2^n} \right)$$



4. Prove that

$$\lim_{n \rightarrow \infty} \left(\frac{1}{\sqrt{1+n^2}} + \frac{1}{\sqrt{2+n^2}} + \dots + \frac{1}{\sqrt{n+n^2}} \right) = 1$$

OR

Test the convergence of the series

let $\sum_{n=1}^{\infty} a_n =$

$$\frac{1+2}{2^3} + \frac{1+2+3}{3^3} + \frac{1+2+3+4}{4^3} + \dots$$

let $a_n = \frac{1+2}{2^3} + \frac{1+2+3}{3^3} + \frac{1+2+3+4}{4^3} + \dots$

then $a_n = \frac{1+2+3+\dots+n}{n^3} \quad \forall n \geq 1, n \in \mathbb{N}$

$$= \frac{N(n+1)}{2n^3}$$

$\left[\because 1+2+3+\dots+n = \frac{n(n+1)}{2} \right]$

$$= \frac{(n+1)}{2n^2}$$

$$= \frac{N(1+1/n)}{2n^2}$$

$$a_n = \frac{(1+1/n)}{2n}$$

first define the series $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n}$, harmonic series
 $b_n = \frac{1}{n}$

let $b_n = \frac{1}{n}$ and on applying limit comparison test

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \left(\frac{1+1/n}{2n} \right) \cdot \frac{n}{1}$$

$$= \lim_{n \rightarrow \infty} \frac{1}{2} \cdot \left(1 + \frac{1}{n} \right) = \frac{1}{2} = \text{finite} \neq 0$$

$\Rightarrow \sum a_n$ and $\sum b_n$ converge or diverge together.

and $\sum b_n = \sum \frac{1}{n}$ is a harmonic p-series which diverges

Hence $\sum a_n$ also diverges.

define the statement.

5. Gauss's Test

Let $\sum u_n$ be a series of positive real numbers and let

$$\frac{u_n}{u_{n+1}} = 1 + \frac{a}{n} + \frac{b_n}{n^p}, \quad p > 1$$

where the sequence (b_n) is bounded. Then $\sum u_n$ is convergent if $a > 1$, and divergent if $a \leq 1$.

Using this test, examine the convergence of the series

$$1 + \frac{2^2}{3^2} + \frac{2^2 4^2}{3^2 5^2} + \frac{2^2 4^2 6^2}{3^2 5^2 7^2} + \dots$$

$$\sum u_n = \frac{1}{1^2} + \frac{2^2}{3^2} + \frac{2^2 4^2}{3^2 5^2} + \dots$$

Ignoring the first term,

$$\text{Let } u_n = \frac{2^2 \cdot 4^2 \cdot 6^2 \dots (2n)^2}{3^2 \cdot 5^2 \cdot 7^2 \dots (2n+1)^2}$$

Now, to apply Gauss test, we have to make in $\frac{u_n}{u_{n+1}}$ form

$$u_{n+1} = \frac{2^2 \cdot 4^2 \cdot 6^2 \dots (2n+2)^2}{3^2 \cdot 5^2 \cdot 7^2 \dots (2n+3)^2}$$

$$\text{Now } \frac{u_n}{u_{n+1}} = \frac{2^2 \cdot 4^2 \cdot 6^2 \dots (2n)^2}{3^2 \cdot 5^2 \cdot 7^2 \dots (2n+1)^2} \cdot \frac{3^2 \cdot 5^2 \cdot 7^2 \dots (2n+3)^2}{2^2 \cdot 4^2 \cdot 6^2 \dots (2n+2)^2}$$

$$= \frac{(2n+3)^2}{(2n+2)^2}$$

$$= \frac{4n^2 + 9 + 12n}{4n^2 + 4 + 8n}$$

$$= \frac{\left(1 + \frac{9}{4n^2} + \frac{12}{4n}\right)}{\left(1 + \frac{4}{4n^2} + \frac{8}{4n}\right)} = \frac{\left(1 + \frac{9}{4n^2} + \frac{3}{n}\right)}{\left(1 + \frac{1}{n^2} + \frac{2}{n}\right)}$$

See Next page!

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$$u_1 = 1, \quad u_2 = \frac{2^2}{3^2}, \quad u_3 = \frac{2^2 \cdot 4^2}{3^2 \cdot 5^2}, \quad u_4 = \frac{2^2 \cdot 4^2 \cdot 6^2}{3^2 \cdot 5^2 \cdot 7^2}, \quad u_5 = \frac{2^2 \cdot 4^2 \cdot 6^2 \cdot 8^2}{3^2 \cdot 5^2 \cdot 7^2 \cdot 9^2}$$

W.H.T.

$$u_1 = \frac{2^2}{3^2}$$

$$u_1 = 1$$

from your nth term

$$\frac{u_n}{u_{n+1}} = \left(\frac{2n}{2n+1} \right)^2 = \frac{4}{\dots}$$

$$\frac{u_{n+1}}{u_n} = \left(\frac{2n}{2n+1} \right)^2$$

$$\frac{4n^2 + 1 + 4n}{4n^2}$$

$$\Rightarrow \frac{u_n}{u_{n+1}} =$$

$$1 + \frac{1}{n} + \frac{1}{4n^2}$$

$$a=1$$

$$b_n = \frac{1}{4}$$

diverges

$$a=1$$

converges for $n > 1$

$$\left(\frac{1}{n} + \frac{1}{n} + 1 \right)$$

$$\left(\frac{1}{n} + \frac{1}{n} + 1 \right)$$

$$\left(\frac{1}{n} + \frac{1}{n} + 1 \right)$$

$$\left(\frac{1}{n} + \frac{1}{n} + 1 \right)$$