

feedback: can do better, but try to define properly every statement or definition you're using

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REAL ANALYSIS TEST 1

Instructions: Questions 1-4 each contain two parts. Attempt any **three** questions out of the first four, solving **one part** from each. Question 5 is **compulsory**. Each question carries 15 marks. **Total marks: 60.**

1. Show that if a convergent series contains only finitely many negative terms, then it is absolutely convergent.
OR
Show that if a series is conditionally convergent, the series of its positive terms diverges, and the series of its negative terms diverges.

Let $\sum a_n$ be a series which is conditionally convergent imply not absolutely convt.
Again let $\sum a_n = \sum a_n^+ + \sum a_n^-$
where $\sum a_n^+$ be positive term series
define them $\sum a_n^-$ be negative term series

Let us take $\sum a_n = \sum \frac{(-1)^n}{n}$ is conditionally convergent
 $b_n = \max(a_n, 0)$
 $c_n = \max(-a_n, 0)$
or $- \min(a_n, 0)$
then $a_n = b_n - c_n$
and $|a_n| = b_n + c_n$
$$\begin{aligned} \sum a_n &= -1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \dots \\ &= \left(\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \dots\right) + \left(-1 - \frac{1}{3} - \frac{1}{5} - \dots\right) \\ &= \sum_{m=1}^{\infty} \frac{(-1)^{2m}}{2m} + \sum_{m=1}^{\infty} \frac{(-1)^{2m+1}}{2m+1} \\ &= \sum a_m^+ + \sum a_m^- \end{aligned}$$

you cannot use a particular example to prove or show the proof.
Examples are used to disprove a statement.

Prove that: Both $\sum a_m^+$ and $\sum a_m^-$ diverge.

Claim - $\sum b_n$ or $\sum c_n$ one of them converges

Suppose $\sum a_m^+$ converge to finite value
$$\begin{aligned} \sum a_n &= \sum a_n^+ + \sum a_n^- \\ L &= R + \sum a_n^- \end{aligned}$$

$$\sum a_n = \sum b_n - \sum a_n^-$$

what you did is okay but you have to write properly and define which strengthen your solution.

given it is conditionally convergent imply $\sum |a_n|$ diverges

1/ or now $\sum a_n$ i.e. you also don't want.
 Suppose $\sum b_n$ converges i.e. the term series -
 and $\sum a_n$ conditionally converges.
 $\Rightarrow \sum c_n = \sum b_n + \sum a_n$ converges.
 BUT, if both the ind. term series
 converge together then,
 $\sum |a_n| \leq \sum b_n + \sum c_n$ (converges)

$\sum a_n^- = L - R < \infty$ (which is finite)
 which is contradicted to our assumption which $\sum a_n^-$ diverge
 Hence our assumption is wrong.
 Both $\sum a_n^+$ and $\sum a_n^-$ diverge.

This is the contradiction.
 As $\sum |a_n|$ diverges because $\sum a_n$ converges conditionally.

Solution was
 Kind OKAY
 for that

$$\frac{3}{15}$$

dth a
 1 i
 $\sum$

3. Let (s_n) be a sequence of nonnegative numbers, and for each n define

$$\sigma_n = \frac{1}{n}(s_1 + s_2 + \dots + s_n)$$

- (a) Show that if $\lim s_n$ exists, then $\lim \sigma_n$ exists and $\lim \sigma_n = \lim s_n$.
 (b) Give an example where $\lim \sigma_n$ exists, but $\lim s_n$ does not exist.

OR

Let $S = \left\{ \frac{1}{2^m} + \frac{1}{2^n} : m, n \in \mathbb{N} \right\}$.

- (a) Show that 0 is a limit point of S .
 (b) If $k \in \mathbb{N}$, show that $\frac{1}{2^k}$ is a limit point of S .
 (c) Find S' (the derived set of S).

Let $S = \left\{ \frac{1}{2^m} + \frac{1}{2^n} : m, n \in \mathbb{N} \right\}$

a) For fixed m and $n \rightarrow \infty$,

$$\frac{1}{2^m} + \frac{1}{2^n} \rightarrow \frac{1}{2^m}$$

so $\frac{1}{2}, \frac{1}{4}, \frac{1}{6}$ are limit point

If both $m, n \rightarrow \infty$

$$\lim_{m, n \rightarrow \infty} S = 0$$

so, 0 is the limit point

Hence

c) $S' = \{0\} \cup \left\{ \frac{1}{2^m} : m \in \mathbb{N} \right\}$

b) $k \in \mathbb{N}$

Let $m=k$ and $n \rightarrow \infty$ then

$$\frac{1}{2^k} + \frac{1}{2^n} \in S$$

$\frac{1}{6}??$

this was required in (a) part.
 But how it is possible
 state the theorem used.
Archimedian property

Σ -definition.

Solution is not complete.

As $n \rightarrow \infty$

$$\frac{1}{2^n} \rightarrow 0$$

$$\therefore S = \frac{1}{2^k} + \frac{1}{2^n} \rightarrow \frac{1}{2^k}$$

$\therefore$ For every $k \in \mathbb{N}$, $\frac{1}{2^k}$ is limit point of S .

for solving it using Gauss trick

4. Prove that

$$\lim_{n \rightarrow \infty} \left(\frac{1}{\sqrt{1+n^2}} + \frac{1}{\sqrt{2+n^2}} + \dots + \frac{1}{\sqrt{n+n^2}} \right) = 1$$

OR

Test the convergence of the series

$$\frac{1+2}{2^3} + \frac{1+2+3}{3^3} + \frac{1+2+3+4}{4^3} + \dots$$

Let $a_n = \frac{1+2+3+\dots+n}{n^3} = \frac{n(n+1)}{2n^3}$ ✓

Also $a_{n+1} = \frac{(n+1)(n+2)}{2(n+1)^3}$ ✓

Now,

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)(n+2)}{2(n+1)^3} \times \frac{2n^3}{n(n+1)}$$

$$= \frac{(1+\frac{1}{n})(1+\frac{2}{n})}{2(1+\frac{1}{n})^3} \times \frac{2}{(1+\frac{1}{n})}$$

$$= \frac{1+\frac{2}{n}}{(1+\frac{1}{n})^3}$$

$\therefore \lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = 0$ ✓

Ratio test fails.

Now $\frac{a_{n+1}}{a_n} = \frac{(1+\frac{2}{n})(1-\frac{1}{n})}{(1+\frac{1}{n})^3}$

Now $\frac{a_n}{a_{n+1}} = \frac{(1+\frac{1}{n})^3}{(1+\frac{2}{n})(1-\frac{1}{n})} = (1+\frac{3}{n} + \frac{3}{n^2} + \frac{1}{n^3}) (1-\frac{2}{n} + \frac{1}{n^2} + \dots)$

you should have compared it with $\sum b_n = \sum \frac{1}{n}$ (harmonic series).
it diverges as a result of p-series test.

✓ $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \frac{(n+1)/2n^2}{1/n} = \frac{1}{2}$
converges and diverges together $\frac{1}{2} \neq 0$ finite.

But Remember

(State comparison test properly)

$$= 1 - \frac{2}{n} + \frac{4}{n^2} + \frac{3}{n} - \frac{6}{n^2} + \dots$$

$$= 1 + \frac{1}{n} - \frac{2}{n^2} + \dots$$

∴ By Gauss test

$$a_n = +1$$

∴ The series a_n diverge.

✓
You did prove it
diverges using Gauss's
test, but in solution you must
state it properly.

Gauss Test

If it is in paper, then give
a reference of the question.

5. Gauss's Test

Let $\{u_n\}$ be a series of positive real numbers and let

$$\frac{u_n}{u_{n+1}} = 1 + \frac{a}{n} + \frac{b_n}{n^p}, \quad p > 1,$$

where the sequence $\{b_n\}$ is bounded. Then $\{u_n\}$ is convergent if $a > 1$, and divergent if $a \leq 1$.

Using this test, examine the convergence of the series

$$1 + \frac{2^2}{3^2} + \frac{2^2 4^2}{3^2 5^2} + \frac{2^2 4^2 6^2}{3^2 5^2 7^2} + \dots$$

Soln.

By neglecting 1st term ✓

$$\text{Let } a_n = \frac{2^2 \cdot 4^2 \cdot 6^2 \dots (2n)^2}{3^2 \cdot 5^2 \cdot 7^2 \dots (2n+1)^2}$$

$$a_{n+1} = \frac{2^2 \cdot 4^2 \cdot 6^2 \dots (2n+2)^2}{3^2 \cdot 5^2 \cdot 7^2 \dots (2n+3)^2}$$

Now

$$\frac{a_{n+1}}{a_n} = \frac{2^2 \cdot 4^2 \dots (2n+2)^2}{3^2 \cdot 5^2 \dots (2n+3)^2} \times \frac{3^2 \cdot 5^2 \dots (2n+1)^2}{2^2 \cdot 4^2 \dots (2n)^2}$$

$$= \frac{\left(1 + \frac{2}{2n}\right)^2 \left(1 + \frac{1}{2n}\right)^2}{\left(1 + \frac{3}{2n}\right)^2}$$

$$\therefore \lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n}$$

$$\frac{a_n}{a_{n+1}} = \frac{(1 + \frac{3}{2n})^2}{(1 + \frac{1}{n})^2 (1 + \frac{1}{2n})^2}$$

$$= \left(1 + \frac{3}{n} + \frac{9}{4n^2}\right) \left(1 - \frac{2}{n} + \frac{3}{n^2} + \dots\right) \left(1 - \frac{1}{n} + \frac{3}{2n^2} + \dots\right)$$

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Define 'h' term
M.I.
and state that
neglecting the
finite number of
terms from series
(convergence of divergence
doesn't get affected).

$$= \left(1 - \frac{2}{n} + \frac{3}{n^2} + \frac{3}{n} - \frac{6}{n^2} + \dots\right) \left(1 - \frac{1}{n} + \frac{3}{2n^2} + \dots\right)$$

$$= \left(1 + \frac{1}{n} + \frac{3}{n^2} + \dots\right) \left(1 - \frac{1}{n} + \frac{3}{2n^2} + \dots\right)$$

$$= \left(1 - \frac{1}{n} + \frac{3}{2n^2} + \frac{1}{n} - \frac{1}{n^2} + \dots\right)$$

$$= \left(1 - \frac{0}{n} + \frac{1}{n^2} + \dots\right)$$

∴ By Gauss test

$$a = 0 < 1$$

∴ The series a_n is divergent.

$b_n = ?$

show that b_n is bounded.